

Groupoids and inverse semigroups associated to W^* -algebras

Anatol Odziejewicz, Aneta Slizewska
 Institute of Mathematics
 University in Białystok
 Akademicka 2, 15-267 Białystok, Poland

February 2, 2012

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Abstract

We investigate the Banach Lie groupoids and inverse semigroups naturally associated to W^* -algebras. We also present statements describing the relationship between these groupoids and the Banach Poisson geometry which follows in the canonical way from the W^* -algebra structure.

1 Introduction

During recent decades the notion of groupoid entered many branches of mathematics including topology [4], differential geometry in general [13] and Poisson geometry [5] in particular as well as the theory of operator algebras [16]. Let us recall shortly that a groupoid is a small category all of whose morphisms

are invertible. In accordance with [13] they are “the natural formulation of a symmetry for objects which have bundle structure”. Nevertheless the role of groupoids is not so widely accepted as that of groups. On the other hand the theory of W^* -algebras (von Neumann algebras) occupies an outstanding place in mathematics and mathematical physics [18].

Motivated by the existence of the canonically defined Banach Lie-Poisson structure on the predual \mathfrak{M}_* of any W^* -algebra \mathfrak{M} , see [14], and by the importance of this structure in the theory of infinite dimensional Hamiltonian systems, see [15], we clarify here some natural connections between Banach Poisson geometry and groupoid theory from one side and W^* -algebras from the other.

In Section 2 we show that the structure of any W^* -algebra \mathfrak{M} naturally defines two important groupoids $\mathcal{U}(\mathfrak{M})$ and $\mathcal{G}(\mathfrak{M})$ the first of which consists of the partial isometries in \mathfrak{M} and the second, being the complexification of $\mathcal{U}(\mathfrak{M})$, consists of the partially invertible elements of \mathfrak{M} . In this section we also discuss canonical actions of $\mathcal{G}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M})$ on the lattice of projections $\mathcal{L}(\mathfrak{M})$ and on the cone \mathfrak{M}_*^+ of the positive normal states on \mathfrak{M} . In the Theorem 2.9 we show that the action groupoid $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ has the faithful representation on the GNS bundle $\bar{\mathbb{E}} \rightarrow \mathfrak{M}_*^+$. The Theorem 2.10 shows that one can consider $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ as a subgroupoid of the groupoid of partial isometries $\mathcal{U}(\bar{\rho}(\mathfrak{M})')$ for the commutant $\bar{\rho}(\mathfrak{M})'$ of the W^* -representation $\bar{\rho} : \mathfrak{M} \rightarrow L^\infty(L^2\Gamma(\mathbb{E}, \mathfrak{M}_*^+))$ of \mathfrak{M} in the Hilbert space of the square-summable sections for the bundle $\bar{\mathbb{E}} \rightarrow \mathfrak{M}_*^+$.

In Section 3 we study inverse semigroups which are the subsets of the groupoid $\mathcal{U}(\mathfrak{M})$ for various type of W^* -algebras. Such inverse semigroups form a class of subgroupoids of $\mathcal{U}(\mathfrak{M})$. In particular we describe the inverse semigroups related to matrix unit inverse semigroups, see Theorem 3.4, the Clifford inverse semigroups (Proposition 3.18), and the monogenic inverse semigroups (Theorem 3.10). Finally we discuss the CAR inverse semigroup given by the canonical anticommutation relations which has fundamental meaning in quantum physics.

The topology and Banach manifold structure of $\mathcal{G}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M})$ are described in Section 4. We show here that $\mathcal{G}(\mathfrak{M})$ is not a topological groupoid with respect to any natural topology of \mathfrak{M} . However the groupoid $\mathcal{U}(\mathfrak{M})$ is a topological groupoid with respect to the uniform topology as well as the $s^*(\mathcal{U}(\mathfrak{M}), \mathfrak{M}_*)$ -topology. Theorem 4.3 describes the topological structure of the action groupoids related to $\mathcal{U}(\mathfrak{M})$. We investigate the complex Banach manifold structure on the lattice $\mathcal{L}(\mathfrak{M})$ and the groupoid $\mathcal{G}(\mathfrak{M})$ and show that $\mathcal{G}(\mathfrak{M})$ is a Banach-Lie groupoid with $\mathcal{L}(\mathfrak{M})$ as its base manifold, see Theorem 4.5. The last statement is also true for the groupoid $\mathcal{U}(\mathfrak{M})$ when we consider it as a real Banach manifold, and the groupoid $\mathcal{G}(\mathfrak{M})$ is the complexification of $\mathcal{U}(\mathfrak{M})$, see Theorem 4.6.

In Section 5 we present several, essential in the present context, statements describing relationship between groupoids $\mathcal{G}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M})$ and the canonical Poisson structure defined on the Banach vector bundles $\mathcal{A}_*\mathcal{G}(\mathfrak{M})$ and $\mathcal{A}_*\mathcal{U}(\mathfrak{M})$ predual to the algebroids $\mathcal{A}\mathcal{G}(\mathfrak{M})$ and $\mathcal{A}\mathcal{U}(\mathfrak{M})$ respectively. From that we conclude that in the framework of W^* -algebras theory there exists the natural illustration of the deep ideas connecting finite dimensional Poisson geometry

and Lie groupoids theory which was investigated in [11], [13], [20].

2 Groupoids associated to W^* -algebras and their representations

In this section we introduce various groupoids defined in a canonical way by the given W^* -algebra \mathfrak{M} . We will also describe representations of these groupoids on vector bundles related to the algebra \mathfrak{M} as well as to its predual \mathfrak{M}_* . The basic facts from the theory of groupoids one can find in appendix to this paper. The detailed presentation of the groupoid theory one can find in [13]. The part of the theory of W^* -algebras indispensable for the subsequent investigations is given in [18] and [19].

By $\mathcal{U}(\mathfrak{M})$ we shall denote the set of all partial isometries in \mathfrak{M} , i.e. $u \in \mathcal{U}(\mathfrak{M})$ if and only if $u^*u \in \mathcal{L}(\mathfrak{M})$, where $\mathcal{L}(\mathfrak{M})$ is the lattice of orthogonal projections $p = p^* = p^2 \in \mathfrak{M}$. For $x \in \mathfrak{M}$ one defines the left support $l(x) \in \mathcal{L}(\mathfrak{M})$ (respectively right support $r(x) \in \mathcal{L}(\mathfrak{M})$) as the smallest projection in \mathfrak{M} such that $l(x)x = x$ (respectively $x r(x) = x$). If $x = x^*$ then $l(x) = r(x) =: s(x)$ and one calls $s(x)$ the support of x . Let

$$x = u|x| \quad (2.1)$$

be the polar decomposition of x , where $u \in \mathcal{U}(\mathfrak{M})$ and $|x| \in \mathfrak{M}^+ := \{x \in \mathfrak{M} : x^* = x > 0\}$, see [18]. Then one has

$$\begin{aligned} l(x) &= s(|x^*|) = uu^*, \\ r(x) &= s(|x|) = u^*u. \end{aligned} \quad (2.2)$$

In this paper we will denote by $G(p\mathfrak{M}p)$ the group of all invertible elements of the W^* -subalgebra $p\mathfrak{M}p \subset \mathfrak{M}$. In particular if $p = 1$ then $G(\mathfrak{M})$ will be the group of all invertible elements of \mathfrak{M} . Similarly by $U(p\mathfrak{M}p)$ and $U(\mathfrak{M})$ we will denote the groups of unitary elements of $p\mathfrak{M}p$ and \mathfrak{M} .

For any $x \in \mathfrak{M}$ one has $|x| \in p\mathfrak{M}p$, where $p = s(|x|)$. Let us define the subset $\mathcal{G}(\mathfrak{M}) \subset \mathfrak{M}$ by

$$\mathcal{G}(\mathfrak{M}) := \{x \in \mathfrak{M} : |x| \in G(p\mathfrak{M}p), \text{ where } p = s(|x|)\}. \quad (2.3)$$

Equivalently, x belongs to $\mathcal{G}(\mathfrak{M})$ if the left multiplication $L_{|x|}$ by $|x|$ defines a right \mathfrak{M} -module isomorphism

$$L_{|x|} : p\mathfrak{M} \xrightarrow{\sim} p\mathfrak{M} \quad (2.4)$$

of the right ideal $p\mathfrak{M}$.

Proposition 2.1. *The subset $\mathcal{G}(\mathfrak{M}) \subset \mathfrak{M}$ has the canonical structure of a groupoid with $\mathcal{L}(\mathfrak{M})$ as the base set. The groupoid structure of $\mathcal{G}(\mathfrak{M})$ is defined as follows:*

(i) the identity section $\varepsilon : \mathcal{L}(\mathfrak{M}) \hookrightarrow \mathcal{G}(\mathfrak{M})$ is the inclusion;

(ii) the source and target maps: $\mathbf{s}, \mathbf{t} : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ are defined by

$$\mathbf{s}(x) := r(x) = u^*u \quad \text{and} \quad \mathbf{t}(x) := l(x) = uu^*; \quad (2.5)$$

(iii) the product

$$\mathcal{G}(\mathfrak{M})^{(2)} \ni (x, y) \mapsto xy \in \mathcal{G}(\mathfrak{M}) \quad (2.6)$$

on the set of composable pairs

$$\mathcal{G}(\mathfrak{M})^{(2)} := \{(x, y) \in \mathcal{G}(\mathfrak{M}) \times \mathcal{G}(\mathfrak{M}); \quad \mathbf{s}(x) = \mathbf{t}(y)\}$$

is given by the product in the W^* -algebra \mathfrak{M} ;

(iv) the inverse map $\iota : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M})$ is given by

$$\iota(x) := |x|^{-1}u^*, \quad (2.7)$$

where u and $|x|$ are defined in the unique way by the polar decomposition (2.1).

Proof. Since $\varepsilon : \mathcal{L}(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M})$ is inclusion the maps $\mathbf{t} : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ and $\mathbf{s} : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ are surjective.

From $xs(x) = x$ one has $yx\mathbf{s}(x) = yx$. This gives $\mathbf{s}(yx) \leq \mathbf{s}(x)$, where " \leq " means lattice order in $\mathcal{L}(\mathfrak{M})$. Using $r(y) = \iota(y)y = l(x)$ and $\iota(y)yx \mathbf{s}(yx) = \iota(y)yx$ we obtain $x \mathbf{s}(yx) = x$. Thus we have $\mathbf{s}(x) \leq \mathbf{s}(yx)$. This shows that $r(yx) = \mathbf{s}(yx) = \mathbf{s}(x) = r(x)$. In a similar way we show that $l(yx) = l(y)$.

The associativity of the product (2.6) follows from the associativity of the algebra product.

Using (2.1) and (2.2) we get

$$\begin{aligned} \iota(x)x &= \varepsilon(\mathbf{s}(x)), \\ x\iota(x) &= \varepsilon(\mathbf{t}(x)), \\ \mathbf{s}(x) &= \mathbf{t}(\iota(x)), \\ \mathbf{t}(x) &= \mathbf{s}(\iota(x)) \end{aligned} \quad (2.8)$$

for $x \in \mathcal{G}(\mathfrak{M})$. The above proves the groupoid structure of $\mathcal{G}(\mathfrak{M})$. \square

From now on we will call $\mathcal{G}(\mathfrak{M})$ the groupoid of partially invertible elements of the W^* -algebra \mathfrak{M} .

Since

$$|x^*| = u|x|u^* \quad (2.9)$$

we see that the groupoid $\mathcal{G}(\mathfrak{M})$ is invariant with respect to $*$ -involution. Thus from the definition of the inverse map $\iota : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M})$ follows that the involution $J : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M})$ defined by

$$J(x) := \iota(x)^* = \iota(x^*) \quad (2.10)$$

is an automorphism of the groupoid $\mathcal{G}(\mathfrak{M})$. We note also that the set of fixed points of $J : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M})$, i.e.

$$\{u \in \mathcal{G}(\mathfrak{M}) : J(u) = u\} \quad (2.11)$$

is the set $\mathcal{U}(\mathfrak{M})$ of all partial isometries of the W^* -algebra \mathfrak{M} . From the above we conclude the following

Proposition 2.2. *The set $\mathcal{U}(\mathfrak{M})$ of partial isometries in \mathfrak{M} is a wide subgroupoid of the groupoid $\mathcal{G}(\mathfrak{M})$.*

The other important wide subgroupoid of $\mathcal{G}(\mathfrak{M})$ is the inner subgroupoid $\mathcal{J}(\mathfrak{M}) \subset \mathcal{G}(\mathfrak{M})$ defined by

$$\mathcal{J}(\mathfrak{M}) := \bigcup_{p \in \mathcal{L}(\mathfrak{M})} (\mathbf{s}^{-1}(p) \cap \mathbf{t}^{-1}(p)). \quad (2.12)$$

It is a totally intransitive subgroupoid and one can consider it as a bundle $s : \mathcal{J}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ of groups $\mathbf{s}^{-1}(p) \cap \mathbf{t}^{-1}(p) = G(p\mathfrak{M}p)$ indexed by $p \in \mathcal{L}(\mathfrak{M})$. One has also the action $I : \mathcal{G}(\mathfrak{M}) * \mathcal{J}(\mathfrak{M}) \rightarrow \mathcal{J}(\mathfrak{M})$ of $\mathcal{G}(\mathfrak{M})$ on $\mathcal{J}(\mathfrak{M})$ defined by

$$I_x y := xy \iota(x) \quad (2.13)$$

for $(x, y) \in \mathcal{G}(\mathfrak{M}) * \mathcal{J}(\mathfrak{M}) := \{(x, y) \in \mathcal{G}(\mathfrak{M}) \times \mathcal{J}(\mathfrak{M}) : r(x) = s(y)\}$. This action is called the inner action. Note that the moment map for the inner action is the support map $s : \mathcal{J}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$. Since for $y \in \mathcal{J}(\mathfrak{M})$ one has $s(y) = l(y) = r(y)$ one can consider the lattice of projections $\mathcal{L}(\mathfrak{M})$ as a subgroupoid of $\mathcal{J}(\mathfrak{M})$ invariant with respect to the action $I : \mathcal{G}(\mathfrak{M}) * \mathcal{J}(\mathfrak{M}) \rightarrow \mathcal{J}(\mathfrak{M})$. So, the inner action (2.13) defines the action $I : \mathcal{G}(\mathfrak{M}) * \mathcal{L}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ of the groupoid $\mathcal{G}(\mathfrak{M})$ on the lattice $\mathcal{L}(\mathfrak{M})$. The moment map for this action is the identity map $id : \mathcal{L}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$.

The groupoid structure of $\mathcal{G}(\mathfrak{M})$ allows us to define the principal bundles:

$$\begin{aligned} \mathbf{s} : \mathbf{t}^{-1}(p) &\rightarrow \mathcal{O}_p \\ \mathbf{t} : \mathbf{s}^{-1}(p) &\rightarrow \mathcal{O}_p \end{aligned} \quad (2.14)$$

over the orbit $\mathcal{O}_p := \{xp\iota(x) : x \in \mathbf{s}^{-1}(p)\}$ of the inner action $I : \mathcal{G}(\mathfrak{M}) * \mathcal{L}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ of $\mathcal{G}(\mathfrak{M})$ on the lattice $\mathcal{L}(\mathfrak{M})$. The structural group for the principal bundles (2.14) is the group $G(p\mathfrak{M}p)$.

The inner action (2.13) defines the equivalence relation on $\mathcal{L}(\mathfrak{M})$:

$$p \sim q \quad \text{iff} \quad q \in \mathcal{O}_p, \quad (2.15)$$

for which the equivalence class $[p]$ is the orbit \mathcal{O}_p generated from the projection $p \in \mathcal{L}(\mathfrak{M})$. Let $\mathcal{L}(p) := \{q \in \mathcal{L}(\mathfrak{M}) : q \leq p\} \subset \mathcal{L}(\mathfrak{M})$ be the lattice ideal of the subprojections of the projection $p \in \mathcal{L}(\mathfrak{M})$. One has the canonically defined relation \prec on the set of equivalence classes of the equivalence relation (2.15), i.e.

$$[q] \prec [p] \quad \text{iff} \quad \bigcup_{q' \in [q]} \mathcal{L}(q') \subset \bigcup_{p' \in [p]} \mathcal{L}(p'). \quad (2.16)$$

From Proposition 2.2 we conclude

Corrolary 2.3. *The inner actions of $\mathcal{G}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M})$ on $\mathcal{L}(\mathfrak{M})$ have the same orbits.*

Therefore one can define the equivalence relation (2.15) and the relation (2.16) taking $\mathcal{U}(\mathfrak{M})$ instead of $\mathcal{G}(\mathfrak{M})$.

Proposition 2.4. *The relation \prec defined in (2.16) is a order relation on the set of the inner action orbits of groupoid $\mathcal{G}(\mathfrak{M})$ on $\mathcal{L}(\mathfrak{M})$. If \mathfrak{M} is a factor then this order is linear.*

Proof. Let us denote by $\max[p]$ the set of maximal elements of the orbit \mathcal{O}_p . Since $\mathcal{L}(p) \subset \mathcal{L}(q)$ iff $p \leq q$ we have $\bigcup_{p' \in \max[p]} \mathcal{L}(p') = \bigcup_{p' \in [p]} \mathcal{L}(p')$. Thus $[p] \prec [q]$ and $[q] \prec [p]$ iff $\bigcup_{p' \in [p]} \mathcal{L}(p') = \bigcup_{q' \in [q]} \mathcal{L}(q')$. From the above it follows that $p' \in \mathcal{L}(q')$ for some $q' \in \max[q]$. Since $p' \in [p]$ and $q' \in [q]$ are maximal elements of the orbits we obtain $[p] = [q]$, i.e. the relation \prec is antisymmetric. The proof of reflexivity and transitivity of the relation \prec is trivial. In the factor case the linearity of order relation \prec follows from Comparability Theorem, e.g. see [18], [19]. \square

The equivalence relation (2.15) is fundamental for the classification of W^* -algebras, see [18], [19]. So, the problem of classification of $\mathcal{U}(\mathfrak{M})$ -orbits on $\mathcal{L}(\mathfrak{M})$ is strictly related to the Murray and von Neumann classification of W^* -algebras. The reason is that the inner action $I : \mathcal{U}(\mathfrak{M}) * \mathcal{L}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ preserves the lattice structure of $\mathcal{L}(\mathfrak{M})$, i.e. for any $(u, p) \in \mathcal{U}(\mathfrak{M}) * \mathcal{L}(\mathfrak{M})$ the maps

$$I_u : \mathcal{L}(p) \rightarrow \mathcal{L}(upu^*) \quad (2.17)$$

are isomorphisms of the lattice ideals. In particular if the projection $z \in \mathcal{L}(\mathfrak{M}) \cap \mathcal{Z}(\mathfrak{M})$ is central, where $\mathcal{Z}(\mathfrak{M})$ is the center of \mathfrak{M} , then the lattice $\mathcal{L}(z) = \mathcal{L}(z\mathfrak{M})$ is preserved by the inner action. This allows us to reduce the classification of $\mathcal{U}(\mathfrak{M})$ -orbits on $\mathcal{L}(\mathfrak{M})$ to the classification of $\mathcal{U}(\mathfrak{M})$ -orbits for the case when \mathfrak{M} is a factor.

In the subsequent part of this section we investigate the representations of the groupoids on the vector bundles which are given by the structure of the W^* -algebra.

Let us begin by briefly explaining that what one understands by representation of the groupoid is a direct generalization of the notion of group representation in the vector space. However for groupoids one takes a vector bundle instead of the vector space. For the purposes of this paper as a rule we assume that the fibres $\pi^{-1}(m)$, $m \in M$, of vector bundle $(\mathbb{E}, M, \pi : \mathbb{E} \rightarrow M)$ under consideration will be not necessary isomorphic. In consequence of that the structural groupoid $\mathcal{G}(\mathbb{E})$ of this bundle would be not necessary transitive on base M .

Recall, see also [13], that the structural groupoid $\mathcal{G}(\mathbb{E})$ consists of linear isomorphisms $e_m^n : \mathbb{E}_m \xrightarrow{\sim} \mathbb{E}_n$ between the fibers of $\pi : \mathbb{E} \rightarrow M$. The base of $\mathcal{G}(\mathbb{E})$ is the base set M of the bundle. The source map $\mathbf{s} : \mathcal{G}(\mathbb{E}) \rightarrow M$ and the target map $\mathbf{t} : \mathcal{G}(\mathbb{E}) \rightarrow M$ are defined by $\mathbf{s}(e_m^n) := m$ and $\mathbf{t}(e_m^n) := n$

respectively. The inverse map is given by $\iota(e_m^n) := (e_m^n)^{-1}$ and the identity section by $\varepsilon(m) := id_m^m$. Finally the product of isomorphisms $e_l^m : \mathbb{E}_l \xrightarrow{\sim} \mathbb{E}_m$ and $e_m^n : \mathbb{E}_m \xrightarrow{\sim} \mathbb{E}_n$ is given by their composition $e_m^n \circ e_l^m : \mathbb{E}_l \xrightarrow{\sim} \mathbb{E}_n$.

Usually one investigates vector bundles with some additional structures. Further we will consider cases when the fibres of $\pi : \mathbb{E} \rightarrow M$ will be provided with these structures. For example such as: Hilbert space structure, W^* -algebra structure, lattice structure or W^* -algebra module structure.

Let G be a groupoid with base set B . One defines the representation of G on the vector bundle $\pi : \mathbb{E} \rightarrow M$ as a groupoid morphism:

$$\begin{array}{ccc} G & \xrightarrow{\phi} & \mathcal{G}(\mathbb{E}) \\ \begin{array}{c} \text{s} \downarrow \downarrow \text{t} \end{array} & & \begin{array}{c} \text{s} \downarrow \downarrow \text{t} \end{array} \\ B & \xrightarrow{\varphi} & M \end{array} \quad (2.18)$$

of G into the structural groupoid $\mathcal{G}(\mathbb{E})$ of the bundle.

After these preliminary remarks let us consider the following two actions of $\mathcal{G}(\mathfrak{M})$ on the W^* -algebra \mathfrak{M} :

- (i) the left action $L : \mathcal{G}(\mathfrak{M}) *_l \mathfrak{M} \rightarrow \mathfrak{M}$ defined by

$$L_x y := xy \quad (2.19)$$

for $(x, y) \in \mathcal{G}(\mathfrak{M}) *_l \mathfrak{M} := \{(x, y) \in \mathcal{G}(\mathfrak{M}) \times \mathfrak{M}; \quad r(x) = l(y)\}$;

- (ii) the right action $R : \mathcal{G}(\mathfrak{M}) *_r \mathfrak{M} \rightarrow \mathfrak{M}$ defined by

$$R_x y := yx \quad (2.20)$$

for $(x, y) \in \mathcal{G}(\mathfrak{M}) *_r \mathfrak{M} := \{(x, y) \in \mathcal{G}(\mathfrak{M}) \times \mathfrak{M}; \quad l(x) = r(y)\}$.

The moment map $\mu : \mathfrak{M} \rightarrow \mathcal{L}(\mathfrak{M})$ (see Appendix C) for the left action L (the right action R) is the left support map $\mu := l : \mathfrak{M} \rightarrow \mathcal{L}(\mathfrak{M})$ (the right support map $\mu := r : \mathfrak{M} \rightarrow \mathcal{L}(\mathfrak{M})$) defined in (2.2). Since the both actions are intertwined by the inverse map, i.e.

$$\iota \circ L_x = R_{\iota(x)} \circ \iota \quad (2.21)$$

we will restrict ourselves to the left action only. All statements concerning the right action $R : \mathcal{G}(\mathfrak{M}) *_r \mathfrak{M} \rightarrow \mathfrak{M}$ we obtain converting statements for the left action L by (2.21).

Let us take $p, q \in \mathcal{L}(\mathfrak{M})$. According to the commonly accepted notation by $\mathcal{G}(\mathfrak{M})_p^q$ we denote the set $\mathbf{t}^{-1}(q) \cap \mathbf{s}^{-1}(p)$. For any $p \in \mathcal{L}(\mathfrak{M})$ one has the following inclusions:

$$\begin{aligned} \mathbf{s}^{-1}(p) &\subset r^{-1}(p) \subset \mathfrak{M}p \\ \mathbf{t}^{-1}(p) &\subset l^{-1}(p) \subset p\mathfrak{M} \end{aligned} \quad (2.22)$$

where $\mathfrak{M}p$ ($p\mathfrak{M}$) is the left (right) W^* -ideal generated by p . Note here that $\mathbf{s} = r|_{\mathcal{G}(\mathfrak{M})}$ and $\mathbf{t} = l|_{\mathcal{G}(\mathfrak{M})}$.

Now we consider the bundle $\pi : \mathcal{M}_R(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ of right \mathfrak{M} -modules over the lattice $\mathcal{L}(\mathfrak{M})$ with total space defined by

$$\mathcal{M}_R(\mathfrak{M}) := \{(y, p) \in \mathfrak{M} \times \mathcal{L}(\mathfrak{M}) : p r(y) = r(y)\} \quad (2.23)$$

and bundle map $\pi := pr_2$ as the projection on the second component of the product $\mathfrak{M} \times \mathcal{L}(\mathfrak{M})$. The fibre $\pi^{-1}(p)$ over $p \in \mathcal{L}(\mathfrak{M})$ is the right ideal $p\mathfrak{M}$ of \mathfrak{M} generated by the projection p . Any element $x \in \mathcal{G}(\mathfrak{M})_p^q$ defines by the left multiplication an isomorphism $L_x : p\mathfrak{M} \xrightarrow{\sim} q\mathfrak{M}$ of the right \mathfrak{M} -modules, i. e.

$$L_x(ay) = L_x(a)y \quad (2.24)$$

for $a \in p\mathfrak{M}$ and $y \in \mathfrak{M}$. The \mathfrak{M} -module isomorphisms $L : p\mathfrak{M} \xrightarrow{\sim} q\mathfrak{M}$, where $p, q \in \mathcal{L}(\mathfrak{M})$, form the groupoid $\mathcal{G}(\mathcal{M}_R(\mathfrak{M}))$ of structural isomorphisms of the fibers of the bundle $\pi : \mathcal{M}_R(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$. One can show that $L = L_x$ for some $x \in \mathcal{G}(\mathfrak{M})_p^q$. Thus we have the following statement:

Proposition 2.5. *The structural groupoid $\mathcal{G}(\mathcal{M}_R(\mathfrak{M}))$ of the bundle $\pi : \mathcal{M}_R(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ is isomorphic to $\mathcal{G}(\mathfrak{M})$.*

Replacing $\mathcal{M}_R(\mathfrak{M})$ by $\mathcal{M}_L(\mathfrak{M})$ and the action $x \rightarrow L_x$ by the right action $x \rightarrow R_x$, where $x \in \mathcal{G}(\mathfrak{M})$, we obtain the anti-isomorphism of $\mathcal{G}(\mathfrak{M})$ with $\mathcal{G}(\mathcal{M}_L(\mathfrak{M}))$. Using the above two representations we obtain a representation of $\mathcal{G}(\mathfrak{M})$ on the bundle $\pi : \mathcal{A}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ of the W^* -subalgebras of \mathfrak{M} with total space $\mathcal{A}(\mathfrak{M})$ defined by

$$\mathcal{A}(\mathfrak{M}) := \{(y, p) \in \mathfrak{M} \times \mathcal{L}(\mathfrak{M}) : y \in p\mathfrak{M}p\} \quad (2.25)$$

and the bundle map by $\pi := pr_2$. The morphism $I : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{G}(\mathcal{A}(\mathfrak{M}))$ of $\mathcal{G}(\mathfrak{M})$ into the structural groupoid $\mathcal{G}(\mathcal{A}(\mathfrak{M}))$ of the bundle $\pi : \mathcal{A}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ is defined as follows

$$I_x := R_{\iota(x)} \circ L_x : p\mathfrak{M}p \rightarrow q\mathfrak{M}q, \quad (2.26)$$

where $x \in \mathcal{G}(\mathfrak{M})_p^q$.

Note here that $\mathcal{J}(\mathfrak{M}) \subset \mathcal{A}(\mathfrak{M})$ and the action $I : \mathcal{G}(\mathfrak{M}) * \mathcal{A}(\mathfrak{M}) \rightarrow \mathcal{A}(\mathfrak{M})$ is an extension of the inner action $I : \mathcal{G}(\mathfrak{M}) * \mathcal{J}(\mathfrak{M}) \rightarrow \mathcal{J}(\mathfrak{M})$. For $u \in \mathcal{U}(\mathfrak{M})_p^q$ we find that $I_u : p\mathfrak{M}p \rightarrow q\mathfrak{M}q$ is an isomorphism of W^* -subalgebras of \mathfrak{M} . Thus we have

Proposition 2.6. *The inner action $I : \mathcal{U}(\mathfrak{M}) * \mathcal{A}(\mathfrak{M}) \rightarrow \mathcal{A}(\mathfrak{M})$ of the partial isometries groupoid $\mathcal{U}(\mathfrak{M})$ on $\mathcal{A}(\mathfrak{M})$ preserves the positivity, normality, selfadjointness and the norm of the elements of the fibres of $\mathcal{A}(\mathfrak{M})$, i.e.:*

- (i) $|I_u x| = I_u |x|$,
- (ii) $xx^* = x^*x$ iff $(I_u x)^*(I_u x) = (I_u x)(I_u x)^*$

$$(iii) \quad x = x^* \quad \text{iff} \quad (I_u x)^* = I_u x$$

$$(iv) \quad \|I_u x\| = \|x\|$$

for $(u, x) \in \mathcal{U}(\mathfrak{M}) * \mathcal{A}(\mathfrak{M})$.

Let $\mathfrak{M}^+, \mathfrak{M}^h$ and \mathfrak{M}^n denote the sets of positive, selfadjoint and normal elements of \mathfrak{M} respectively. Let S be the sphere in \mathfrak{M} , i. e. $x \in S$ if and only if $\|x\| = 1$. We conclude from Proposition 2.6 that the subsets $\mathcal{J}(\mathfrak{M}) \cap \mathfrak{M}^+$, $\mathcal{J}(\mathfrak{M}) \cap \mathfrak{M}^h$, $\mathcal{J}(\mathfrak{M}) \cap \mathfrak{M}^n$, $\mathcal{J}(\mathfrak{M}) \cap S$ and $\mathcal{J}(\mathfrak{M}) \cap \mathcal{U}(\mathfrak{M})$ are invariant with respect to the inner action $I : \mathcal{U}(\mathfrak{M}) * \mathcal{J}(\mathfrak{M}) \rightarrow \mathcal{J}(\mathfrak{M})$. Let us also note that the lattice of projections $\mathcal{L}(\mathfrak{M})$ consists of the extreme points in $\mathfrak{M}^+ \cap S$, e.g. see [18].

Theorem 2.7. *The actions $L : \mathcal{U}(\mathfrak{M}) *_l \mathfrak{M} \rightarrow \mathfrak{M}$ and $R : \mathcal{U}(\mathfrak{M}) *_r \mathfrak{M} \rightarrow \mathfrak{M}$ defined by (2.19) and (2.20) are free. Their orbits are indexed by elements of the cone \mathfrak{M}^+ of positive selfadjoint elements of \mathfrak{M} .*

Proof. Since left and right actions are intertwined by the inverse map (2.7) it is enough to consider the case of the left action L . Let us assume that there are elements $u_1, u_2 \in \mathcal{U}(\mathfrak{M})$ such that $y := u_1 x = u_2 x$ for $u_1^* u_1 = v v^* = u_2^* u_2$, where $v \in \mathcal{L}(\mathfrak{M})$ is defined by

$$x = v|x|. \quad (2.27)$$

Since $|y|^2 = y^* y = x^* u_1^* u_1 x = x^* v^* v x = x^* x = |x|^2$ we have $y = u_1 v |y| = u_2 v |y|$. Thus from the uniqueness of the polar decomposition (2.1) we obtain $u_1 v = u_2 v$. The above gives $u_1 = u_1 u_1^* u_1 = u_1 v v^* = u_2 v v^* = u_2 u_2^* u_2 = u_2$. So, the left action L is free.

Taking the polar decomposition $x = v|x|$ of $x \in \mathfrak{M}$ we obtain that $v^* x = v^* v |x| = |x| \in \mathfrak{M}^+$. So any orbit \mathcal{O}_x of $\mathcal{U}(\mathfrak{M})$ intersects \mathfrak{M}^+ . If $x, y \in \mathcal{O}_x \cap \mathfrak{M}^+$ then $x = |x|$, $y = |y|$ and $|y| = u|x|$ for some $u \in \mathcal{U}(\mathfrak{M})$. Thus from uniqueness of the polar decomposition we obtain $x = |x| = |y| = y$. \square

For the sake of completeness and the further applications let us consider the actions

$$\begin{aligned} L_* : \mathcal{U}(\mathfrak{M}) *_l \mathfrak{M}_* &\rightarrow \mathfrak{M}_* \\ R_* : \mathcal{U}(\mathfrak{M}) *_r \mathfrak{M}_* &\rightarrow \mathfrak{M}_* \end{aligned} \quad (2.28)$$

of $\mathcal{U}(\mathfrak{M})$ on \mathfrak{M}_* which are predual to the actions $L : \mathcal{U}(\mathfrak{M}) \times \mathfrak{M} \rightarrow \mathfrak{M}$ and $R : \mathcal{U}(\mathfrak{M}) \times \mathfrak{M} \rightarrow \mathfrak{M}$ respectively.

Referring to [19] we recall, that the left predual action $L_* : \mathfrak{M} \times \mathfrak{M}_* \rightarrow \mathfrak{M}_*$ (respectively the right predual action $R_* : \mathfrak{M} \times \mathfrak{M}_* \rightarrow \mathfrak{M}_*$) of W^* -algebra \mathfrak{M} on the predual Banach space \mathfrak{M}_* is defined by:

$$\langle x, L_{*a}\omega \rangle := \langle xa, \omega \rangle \quad (\text{respectively } \langle x, R_{*a}\omega \rangle := \langle ax, \omega \rangle) \quad (2.29)$$

for any $x \in \mathfrak{M}$, where $a \in \mathfrak{M}$ and $\omega \in \mathfrak{M}_*$. So, one has

$$(L_{*a})^* = R_a, \quad \text{and} \quad (R_{*a})^* = L_a. \quad (2.30)$$

For any element $\omega \in \mathfrak{M}_*$ one takes the closed left invariant subspace $[\mathfrak{M}\omega] \subset \mathfrak{M}_*$ (respectively the right invariant subspace $[\omega\mathfrak{M}] \subset \mathfrak{M}_*$) generated from ω by the left (respectively right) action of \mathfrak{M} (2.29). The annihilator $[\mathfrak{M}\omega]^0 \subset \mathfrak{M}$ of the Banach subspace $[\mathfrak{M}\omega] \subset \mathfrak{M}$ is the right W^* -ideal in \mathfrak{M} . Similarly the annihilator $[\omega\mathfrak{M}]^0 \subset \mathfrak{M}$ of the Banach subspace $[\omega\mathfrak{M}] \subset \mathfrak{M}$ is the left W^* -ideal in \mathfrak{M} . Thus there exist the orthogonal projections $e, f \in \mathfrak{M}$ such that $[\mathfrak{M}\omega]^0 = e\mathfrak{M}$ and $[\omega\mathfrak{M}]^0 = \mathfrak{M}f$. The projection e is the greatest one of all the projections $q \in \mathfrak{M}$ such that $R_{*q}\omega = 0$. Similarly the projection f is the greatest one of all the projections $q \in \mathfrak{M}$ such that $L_{*q}\omega = 0$. Thus one defines the map

$$r_*(\omega) := 1 - e \quad (2.31)$$

and

$$l_*(\omega) := 1 - f, \quad (2.32)$$

where $(1-e)$ and $(1-f)$ are the least projection with the property $R_{*(1-e)}\omega = \omega$ and $L_{*(1-f)}\omega = \omega$ respectively. The projections $r_*(\omega)$ and $l_*(\omega)$ are called, respectively, the right support projection and the left support projection of $\omega \in \mathfrak{M}_*$. It follows from the polar decomposition (e.g. see [19])

$$\omega = L_{*v}|\omega| \quad (2.33)$$

of $\omega \in \mathfrak{M}_*$, where $v \in \mathcal{U}(\mathfrak{M})$ and $|\omega| \in \mathfrak{M}_*^+$, that

$$r_*(\omega) = v^*v \quad \text{and} \quad l_*(\omega) = vv^*.$$

Considering $r_* : \mathfrak{M}_* \rightarrow \mathcal{L}(\mathfrak{M})$ and $l_* : \mathfrak{M}_* \rightarrow \mathcal{L}(\mathfrak{M})$ as moment maps we define the actions (2.28) by

$$\mathcal{U}(\mathfrak{M}) *_{r_*} \mathfrak{M}_* \ni (u, \omega) \mapsto R_{*u}\omega \in \mathfrak{M}_* \quad (2.34)$$

and

$$\mathcal{U}(\mathfrak{M}) *_{l_*} \mathfrak{M}_* \ni (u, \omega) \mapsto L_{*u}\omega \in \mathfrak{M}_*, \quad (2.35)$$

respectively, where

$$\mathcal{U}(\mathfrak{M}) *_{r_*} \mathfrak{M}_* = \{(u, \omega) \in \mathcal{U}(\mathfrak{M}) \times \mathfrak{M}_*; \quad \mathbf{t}(u) = uu^* = r_*(\omega)\}$$

and

$$\mathcal{U}(\mathfrak{M}) *_{l_*} \mathfrak{M}_* = \{(u, \omega) \in \mathcal{U}(\mathfrak{M}) \times \mathfrak{M}_*; \quad \mathbf{s}(u) = u^*u = l_*(\omega)\}.$$

Let us define the bundle $\pi_* : \mathcal{A}_*(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ predual to the bundle of the W^* -algebras $\pi : \mathcal{A}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$. In this case the total space is the following

$$\mathcal{A}_*(\mathfrak{M}) := \{(\omega, p) \in \mathfrak{M}_* \times \mathcal{L}(\mathfrak{M}) : r_*(\omega) = p r_*(\omega) \text{ and } l_*(\omega) = l_*(\omega)p\} \quad (2.36)$$

and the bundle map π_* is the projection of $(\omega, p) \in \mathfrak{M}_* \times \mathcal{L}(\mathfrak{M})$ on the second component. Note that one can identify the fibre $\pi_*^{-1}(p) = (R_{*p} \circ L_{*p})(\mathfrak{M})$, $p \in \mathcal{L}(\mathfrak{M})$, with the Banach space $(p\mathfrak{M}p)_*$ predual to subalgebra $p\mathfrak{M}p$.

We define the predual inner action

$$I_* : \mathcal{G}(\mathfrak{M}) * \mathcal{A}_*(\mathfrak{M}) \rightarrow \mathcal{A}_*(\mathfrak{M}) \quad (2.37)$$

of the groupoid $\mathcal{G}(\mathfrak{M})$ on $\mathcal{A}_*(\mathfrak{M})$ where $\mathcal{G}(\mathfrak{M}) * \mathcal{A}_*(\mathfrak{M}) := \{(x, (\omega, p)) : \mathbf{s}(x) = p\}$ and the bundle map $\pi_* : \mathcal{A}_*(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ is the moment map.

Now we define the following subbundles of $\pi_* : \mathcal{A}_*(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$. The subbundle $\pi_* : \mathcal{J}_*(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ whose total space is defined by

$$\mathcal{J}_*(\mathfrak{M}) := \{(\omega, p) \in \mathcal{A}_*(\mathfrak{M}) : l_*(\omega) = r_*(\omega) = p\}. \quad (2.38)$$

The subbundle of selfadjoint normal functionals $\pi_* : \mathcal{A}_*^h(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ for which the total space is

$$\mathcal{A}_*^h(\mathfrak{M}) := \{(\omega, p) \in \mathcal{A}_*(\mathfrak{M}) : \omega^* = \omega\}. \quad (2.39)$$

The subbundle of positive normal functionals $\pi_* : \mathcal{A}_*^+(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ with the set

$$\mathcal{A}_*^+(\mathfrak{M}) := \{(\omega, p) \in \mathcal{A}_*(\mathfrak{M}) : \omega^* = \omega > 0\}. \quad (2.40)$$

as the total space.

When we restrict the action (2.37) to subgroupoid $\mathcal{U}(\mathfrak{M}) \subset \mathcal{G}(\mathfrak{M})$ we obtain the following statement.

Proposition 2.8. *For the predual inner action $I_* : \mathcal{U}(\mathfrak{M}) * \mathcal{A}_*(\mathfrak{M}) \rightarrow \mathcal{A}_*(\mathfrak{M})$ of the partial isometries groupoid $\mathcal{U}(\mathfrak{M})$ on $\mathcal{A}_*(\mathfrak{M})$ one has:*

$$\omega = \omega^* \text{ iff } (I_{*u}\omega)^* = I_{*u}\omega \quad (2.41)$$

$$\|I_{*u}\omega\| = \|\omega\| \quad (2.42)$$

$$|I_{*u}\omega| = I_{*u}|\omega| \quad (2.43)$$

for $(u, (\omega, p)) \in \mathcal{U}(\mathfrak{M}) * \mathcal{A}_*(\mathfrak{M})$, i.e. the subbundles (2.38), (2.40), (2.39) are invariant with respect to this action.

Proof. In order to prove (2.41) we note that for $\omega \in \pi_*^{-1}(u^*u)$ we have $\langle \omega^*, x \rangle := \langle \omega, x^* \rangle$, where $x \in \mathfrak{M}$, so we obtain

$$(I_{*u}\omega)^* = I_{*u}\omega^*.$$

Thus and from $I_{*u^*u}\omega = \omega$ we have that $\omega = \omega^*$ iff $(I_{*u}\omega)^* = I_{*u}\omega$. Since $\|u\| = 1$ and $L_{*u^*u}\omega = \omega$ one has

$$\|I_{*u}\omega\| \leq \|\omega\| \leq \|L_{*u}\omega\|. \quad (2.44)$$

Similarly we prove that $\|R_{*u}\omega\| = \|\omega\|$. Thus we have (2.42).

Let us assume that $\mathfrak{M}_* \ni \omega \geq 0$ then for any $x \in \mathfrak{M}^+$ one has $u^*xu \in \mathfrak{M}^+$ and

$$\langle I_{*u}\omega, x \rangle = \langle \omega, u^*xu \rangle \geq 0.$$

Thus we find that $I_{*u}\omega \geq 0$ iff $\omega \geq 0$. If $\omega = \omega^*$ then for any $x \in \mathfrak{M}$ we have

$$\langle (I_{*u}\omega)^*, x \rangle = \overline{\langle \omega, u^*xu \rangle} = \langle \omega^*, (u^*xu)^* \rangle = \langle I_{*u}\omega^*, x \rangle.$$

The above shows that inner action commutes with conjugation operation.

Let us take the polar decomposition of $\omega \in \pi_*^{-1}(u^*u)$

$$\omega = L_{*v}|\omega|. \quad (2.45)$$

We note that the polar decomposition of $I_{*u}\omega \in \pi_*^{-1}(u^*u)$, where $v^*v \leq u^*u$ and $vv^* \leq u^*u$, is given by

$$I_{*u}\omega = L_{uvu^*}|I_{*u}\omega|. \quad (2.46)$$

From (2.46) it follows that

$$|I_{*u}\omega| = L_{(uvu^*)^*}I_{*u}\omega.$$

Thus for any $x \in \mathfrak{M}$ we have

$$\begin{aligned} \langle |I_{*u}\omega|, x \rangle &= \langle L_{(uvu^*)^*}I_{*u}\omega, x \rangle = \langle I_{*u}\omega, xuv^*u^* \rangle = \\ &= \langle \omega, u^*xuv^*u^*u \rangle = \langle \omega, u^*xuv^* \rangle = \langle L_{*v^*}\omega, u^*xu \rangle = \langle |\omega|, u^*xu \rangle = \\ &= \langle I_{*u}|\omega|, x \rangle. \end{aligned}$$

Thus we obtain (2.43) □

Now basing on GNS construction we define the pre-Hilbert bundle $\pi : \mathbb{E} \rightarrow \mathfrak{M}_*^+$ over the cone of the positive normal states \mathfrak{M}_*^+ . The total space \mathbb{E} and bundle projection we define as follows

$$\mathbb{E} := \{(x, \omega) \in \mathfrak{M} \times \mathfrak{M}_*^+ : xs_*(\omega) = x\} \quad (2.47)$$

and $\pi := pr_2|_{\mathbb{E}}$.

Since for $\omega \in \mathfrak{M}_*^+$ one has $\mathbb{E}_\omega = \pi^{-1}(\omega) = \mathfrak{M}s_*(\omega)$ the scalar product

$$\mathbb{E}_\omega \times \mathbb{E}_\omega \ni (x, y) \mapsto \langle x|y \rangle_\omega := \langle \omega, x^*y \rangle \in \mathbb{C} \quad (2.48)$$

is non degenerate. Thus it defines the pre-Hilbert space structure on \mathbb{E}_ω . Note here that $\langle \omega, x^*x \rangle = 0$ if and only if $x \in \mathfrak{M}(1 - s_*(\omega))$.

Completing \mathbb{E}_ω with respect to the norm $\|x\|_\omega := \langle \omega, x^*x \rangle^{\frac{1}{2}}$ we obtain the bundle $\bar{\pi} : \bar{\mathbb{E}} \rightarrow \mathfrak{M}_*^+$ of Hilbert spaces. Since for the clear reasons we will call this bundle the GNS bundle.

Theorem 2.9. *One has a faithful representation*

$$\begin{array}{ccc}
\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+ & \xrightarrow{\phi} & \mathcal{G}(\overline{\mathbb{E}}) \\
\begin{array}{c} s \downarrow \downarrow t \\ \mathfrak{M}_*^+ \end{array} & & \begin{array}{c} s \downarrow \downarrow t \\ \mathfrak{M}_*^+ \end{array} \\
& \xrightarrow{id} &
\end{array} \tag{2.49}$$

of the right action of groupoid $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ on the GNS Hilbert spaces bundle $\overline{\pi} : \overline{\mathbb{E}} \rightarrow \mathfrak{M}_*^+$ with the fibres isomorphisms $\phi(u, \omega) : \overline{\mathbb{E}}_\omega \rightarrow \overline{\mathbb{E}}_{I_{*u}\omega}$ defined as follows

$$\phi(u, \omega)(xs_*(\omega), \omega) := (xs_*(\omega)u^*, I_{*u}\omega) \tag{2.50}$$

Proof. The following sequence of equalities

$$\begin{aligned}
& \langle \phi(u, \omega)xs_*(\omega) | \phi(u, \omega)ys_*(\omega) \rangle_{I_{*u}\omega} = \\
& = \langle I_{*u}\omega, (xs_*(\omega)u^*)^*ys_*(\omega)u^* \rangle = \\
& = \langle \omega, u^*us_*(\omega)x^*ys_*(\omega)u^*u \rangle = \\
& = \langle \omega, (xs_*(\omega))^*ys_*(\omega) \rangle = \langle xs_*(\omega) | ys_*(\omega) \rangle_\omega
\end{aligned} \tag{2.51}$$

shows that $\phi(u, \omega) : \overline{\mathbb{E}}_\omega \rightarrow \overline{\mathbb{E}}_{I_{*u}\omega}$ extends to isomorphism of Hilbert spaces.

For elements $(u, \omega), (v, \lambda) \in \mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ such that $\mathbf{t}(v) = \mathbf{s}(u)$, i.e. $\omega = I_{*v}\lambda$ we have

$$\begin{aligned}
\phi((u, \omega)(v, \lambda))(xs_*(\lambda), \lambda) &= \phi(uv, \lambda)(xs_*(\lambda), \lambda) = (xs_*(\lambda)(uv)^*, I_{*uv}\lambda) = \\
&= \phi(u, \omega)(xs_*(\lambda)v^*, I_{*u}\lambda) = (\phi(u, \omega) \circ (\phi(v, \lambda)))(xs_*(\lambda), \lambda)
\end{aligned}$$

for any $(xs_*(\lambda), \lambda) \in \overline{\mathbb{E}}_\lambda$. Thus we obtain

$$\phi((u, \omega)(v, \lambda)) = \phi(u, \omega) \circ \phi(v, \lambda). \tag{2.52}$$

We recall that $(u, \omega)(v, \lambda)$ is the product of $(\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+)^{(2)}$ defined by (6.11).

One can easily check that for $\phi(u, \omega)$ and $\phi(u^*, I_{*u}\omega)$ we have

$$\begin{aligned}
(\phi(u^*, I_{*u}\omega) \circ \phi(u, \omega))(xs_*(\omega), \omega) &= \phi(u^*, I_{*u}\omega)(xs_*(\omega)u^*, I_{*u}\omega) = \\
&= (xs_*(\omega)u^*u, I_{*u^*u}\omega) = (xs_*(\omega), \omega)
\end{aligned}$$

for any $(xs_*(\omega), \omega) \in \overline{\mathbb{E}}_\omega$. The above shows that

$$\phi(u^*, I_{*u}\omega) \circ \phi(u, \omega) = id|_{\overline{\mathbb{E}}_\omega}.$$

In the similar way we prove that

$$\phi(u, \omega) \circ \phi(u^*, I_{*u}\omega) = id|_{\overline{\mathbb{E}}_{I_{*u}\omega}}.$$

Thus we get

$$(\phi(u, \omega))^{-1} = \phi(u^*, I_{*u}\omega). \tag{2.53}$$

For any $(u, \omega) \in \mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ one has

$$\begin{aligned}(id \circ \mathbf{s})(u, \omega) &= id(\omega) = \omega, \\ (id \circ \mathbf{t})(u, \omega) &= id(I_{*u}\omega) = I_{*u}\omega\end{aligned}$$

and

$$\begin{aligned}(\mathbf{s} \circ \phi)(u, \omega)(xs_*(\omega), \omega) &= \mathbf{s}(\phi(u, \omega)(xs_*(\omega), \omega)) = \omega, \\ (\mathbf{t} \circ \phi)(u, \omega)(xs_*(\omega), \omega) &= \mathbf{t}(\phi(u, \omega)(xs_*(\omega), \omega)) = I_{*u}\omega,\end{aligned}$$

which shows that $id \circ \mathbf{s} = \mathbf{s} \circ \phi$ and $id \circ \mathbf{t} = \mathbf{t} \circ \phi$, i. e. the diagram (2.49) is commutative. The above shows that ϕ is a groupoid morphism.

From

$$\phi(u, \omega) = \phi(u', \omega'). \quad (2.54)$$

we find that

$$\begin{aligned}\omega &= \omega' \\ s_*(\omega) &= u^*u = u'^*u' = s_*(\omega')\end{aligned} \quad (2.55)$$

and

$$xs_*(\omega)u^* = xs_*(\omega)u'^* \quad (2.56)$$

for any $x \in \mathfrak{M}$. Setting $x = s_*(\omega)$ in (2.56) we prove that from (2.54) it follows

$$(u, \omega) = (u', \omega').$$

Thus ϕ is the faithful morphism of groupoids. \square

In order to obtain a faithful W^* -representation of \mathfrak{M} , see [18], we recall that \mathbb{E}_ω is a left W^* -ideal of \mathfrak{M} . Hence one has W^* -representation $\overline{\rho}_\omega : \mathfrak{M} \rightarrow L^\infty(\overline{\mathbb{E}_\omega})$ of \mathfrak{M} in the W^* -algebra of bounded operators on Hilbert space $\overline{\mathbb{E}_\omega}$, defined by the continuous extension of

$$\rho_\omega(x)ys_*(\omega) := xys_*(\omega), \quad (2.57)$$

where $x \in \mathfrak{M}$ and $ys_*(\omega) \in \mathbb{E}_\omega$.

Let us denote by $L^2\Gamma(\mathbb{E}, \mathfrak{M}_*^+)$ the Hilbert space of square summable sections $\psi : \mathfrak{M}_*^+ \rightarrow \overline{\mathbb{E}}$

$$\sum_{\omega \in \mathfrak{M}_*^+} \|\psi(\omega)\|^2 < \infty \quad (2.58)$$

of the GNS bundle.

The direct sum

$$\overline{\rho} := \bigoplus_{\omega \in \mathfrak{M}_*^+} \overline{\rho}_\omega \quad (2.59)$$

is a faithful W^* -representation $\overline{\rho} : \mathfrak{M} \rightarrow L^\infty(L^2\Gamma(\mathbb{E}, \mathfrak{M}_*^+))$ of \mathfrak{M} in the Hilbert space $L^2\Gamma(\mathbb{E}, \mathfrak{M}_*^+)$. Recall that the $*$ -homomorphism of W^* -algebras is a W^* -homomorphism if it is a map $\phi : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ continuous with respect to the $\sigma(\mathfrak{M}_1, \mathfrak{M}_{1*})$ -topology and $\sigma(\mathfrak{M}_2, \mathfrak{M}_{2*})$ -topology.

Let $\overline{\rho}(\mathfrak{M})'$ be the commutant of $\overline{\rho}(\mathfrak{M})$ in the operator algebra $L^\infty(L^2\Gamma(\mathbb{E}, \mathfrak{M}_*^+))$.

Theorem 2.10. *There exists a groupoid monomorphism*

$$\Lambda : \mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+ \rightarrow \mathcal{U}(\overline{\rho}(\mathfrak{M})') \quad (2.60)$$

*of the action groupoid $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ into the groupoid of partial isometries $\mathcal{U}(\overline{\rho}(\mathfrak{M})')$ of the W^* -algebra $\overline{\rho}(\mathfrak{M})'$.*

Proof. Any element $e_{\omega_1}^{\omega_2} \in \mathcal{G}(\mathbb{E})_{\omega_1}^{\omega_2} \subset \mathcal{G}(\mathfrak{M})$ of the structural groupoid which maps $e_{\omega_1}^{\omega_2} : \overline{\mathbb{E}}_{\omega_1} \rightarrow \overline{\mathbb{E}}_{\omega_2}$ has the extension to a partial isometry of the Hilbert space $L^2\Gamma(\mathbb{E}, \mathfrak{M}_*^+)$. Since Hilbert subspaces $\overline{\mathbb{E}}_{\omega_1}$ and $\overline{\mathbb{E}}_{\omega_2}$ of $L^2\Gamma(\mathbb{E}, \mathfrak{M}_*^+)$ are invariant with respect to the representation $\overline{\rho}$ we conclude that $e_{\omega_1}^{\omega_2} \in \mathcal{U}(\overline{\rho}(\mathfrak{M})')$. Hence we obtain a monomorphism $\iota : \mathcal{G}(\mathbb{E}) \hookrightarrow \mathcal{U}(\overline{\rho}(\mathfrak{M})')$. Thus Proposition 2.9 implies that $\Lambda = \iota \circ \phi$ is a groupoid monomorphism. \square

Finally let us note that the projection

$$pr_1 : \mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+ \rightarrow \mathcal{U}(\mathfrak{M}) \cong \mathcal{U}(\overline{\rho}(\mathfrak{M})') \quad (2.61)$$

of $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ on the first component of the product $\mathcal{U}(\mathfrak{M}) \times \mathfrak{M}_*^+$ defines a morphism of the groupoids. The map (2.61) is an epimorphism of groupoid if and only if \mathfrak{M} is a σ -finite W^* -algebra.

3 Inverse semigroups in $\mathcal{U}(\mathfrak{M})$

Usually one finds in specialized papers the constructions of the operator algebras from the representation of given inverse semigroup. See for example [16], where also one can find other references. Here we will follow conversely obtaining inverse semigroup from given W^* -algebra.

Let us recall that the **inverse semigroup** is a semigroup S in which for each element $s \in S$ exists a unique element $t \in S$ such that

$$sts = s \quad tst = t. \quad (3.1)$$

The element t is denoted by s^* and the map $s \rightarrow s^*$ is involution on S . An important role plays the subset $E(S) \subset S$ of idempotents of the inverse semigroup S . It's easy to check that for any $s \in S$ elements ss^* and s^*s belong to $E(S)$. For any elements $e, f \in E(S)$ we have $e = e^*$ and $ef = fe$.

It is a well-known fact in the theory of the inverse semigroups, which follows directly from the Vagner-Preston theorem [9], that any inverse semigroup S is composed by the partial isometries in the Hilbert space $l^2(S)$, see [16]. On the other hand any W^* -algebra \mathfrak{M} is also realized by bounded operators in some Hilbert space \mathcal{H} , i.e. $\mathfrak{M} \subset L^\infty(\mathcal{H})$. These facts motivate us to investigate the inverse semigroups which are composed by partial isometries of W^* -algebra, i.e. $S \subset \mathcal{U}(\mathfrak{M})$.

Remark 3.1. The product uv of two partial isometries $u, v \in \mathcal{U}(\mathfrak{M})$ is a partial isometry if and only if the initial projection of u and the final projection of v commute, see [8].

Let us assume that partial isometries $u, v \in S$ satisfy property (3.1). Then one has

$$\begin{aligned} u^* &= u^*uu^* = u^*uvuu^* = u^*uvv^*vv^*vu^* = \\ &= vv^*u^*uvuu^*v^*v = v(v^*(u^*uu^*)v^*)v = v. \end{aligned} \quad (3.2)$$

The equality (3.2) shows that for $S \subset \mathcal{U}(\mathfrak{M})$ the element v inverse to u in sense of (3.1) is unique and it is defined as u^* .

So, let us assume that $\mathcal{S} \subset \mathfrak{M}$. Hence the set E of idempotents of \mathcal{S} is given by $E = \mathbf{s}(\mathcal{S}) = \mathbf{t}(\mathcal{S}) \subset \mathcal{L}(\mathfrak{M})$ and $\mathcal{S} \subset \mathcal{U}(E)$ where

$$\mathcal{U}(E) := \mathbf{s}^{-1}(E) \cap \mathbf{t}^{-1}(E) \quad (3.3)$$

is the full subgroupoid of the groupoid of partial isometries $\mathcal{U}(\mathfrak{M})$. Since the product uv in \mathcal{S} is defined for all $(u, v) \in \mathcal{S} \times \mathcal{S}$ it is defined also for $(u, v) \in \mathcal{S}^{(2)} = \{(u, v) : \mathbf{s}(u) = \mathbf{t}(v)\}$. Therefore, the inverse semigroup $\mathcal{S} \subset \mathcal{U}(E) \subset \mathcal{U}(\mathfrak{M})$ can be considered as a subgroupoid of the groupoid $\mathcal{U}(\mathfrak{M})$. In this context the question when $\mathcal{U}(E)$ is the inverse semigroup arises.

Proposition 3.2. *The subgroupoid $\mathcal{U}(E)$ complemented by $\{0\}$ is an inverse semigroup if and only if:*

- (i) for $p, q \in E$ one has $pq = qp \in E$,
- (ii) for $u \in \mathcal{U}(E)$ and $p \in E$ one has $upu^* \in E$ or $upu^* = 0$.

Proof. Let us assume that $\mathcal{U}(E)$ is an inverse semigroup. Then $E \subset \mathcal{U}(E)$ consists of idempotents of $\mathcal{U}(E)$ and is a commutative subsemigroup of $\mathcal{U}(E)$. If $p \in E$ and $u \in \mathcal{U}(E)$ then $up \in \mathcal{U}(E)$. Thus $upu^* = up(up)^* \in E$.

Now, let us assume that properties (i) and (ii) of the proposition are valid. Then for $u, v \in \mathcal{U}(E)$ we have

$$uv(uv)^*uv = uvv^*u^*uv = uu^*uvv^*v = uv.$$

So, uv is a partial isometry and $uv(uv)^* = uvv^*u^* \in E$. This shows that $uv \in \mathcal{U}(E)$. \square

From the Proposition 3.2 we conclude:

Corollary 3.3. *The partial isometries groupoid $\mathcal{U}(\mathfrak{M})$ is an inverse semigroup if and only if W^* -algebra \mathfrak{M} is abelian.*

Proof. If $\mathcal{U}(\mathfrak{M})$ is an inverse semigroup then the lattice $\mathcal{L}(\mathfrak{M})$ is a Boolean one and $pq = qp$ for $p, q \in \mathcal{L}(\mathfrak{M})$. From this and from the spectral theorem we obtain that \mathfrak{M} is an abelian W^* -algebra. If \mathfrak{M} is abelian and $u, v \in \mathcal{U}(\mathfrak{M})$ then

$$uv(uv)^*uv = uvv^*u^*uv = uu^*uvv^*v = uv.$$

It means that $\mathcal{U}(\mathfrak{M})$ is closed with respect to multiplication, i.e. $uv \in \mathcal{U}(\mathfrak{M})$. So, $\mathcal{U}(\mathfrak{M})$ is an inverse semigroup. \square

Example 3.1. Let us consider a family $\{p_i\}_{i \in I}$ of mutually orthogonal projections $p_i p_j = \delta_{ij} p_i$ in \mathfrak{M} . We define $\mathcal{U}_{ij} := \mathbf{t}^{-1}(p_i) \cap \mathbf{s}^{-1}(p_j)$, i.e. $u_{ij} \in \mathcal{U}_{ij}$ iff $u_{ij}^* u_{ij} = p_j$ and $u_{ij} u_{ij}^* = p_i$. So, for $u_{ij} \in \mathcal{U}_{ij}$ and $u_{kl} \in \mathcal{U}_{kl}$ one has

$$u_{ij} u_{kl} = \delta_{jk} u_{ik} u_{kl} \quad (3.4)$$

and

$$u_{ij}^* \in \mathcal{U}_{ji}. \quad (3.5)$$

In the case when $\mathcal{U}_{ij} = \emptyset$ we will assume in (3.4) and (3.5) by definition that $u_{ij} = 0$. Concluding we see that $\mathcal{U}(E) \cup \{0\}$, where $E = \{p_i\}_{i \in I}$, is an inverse semigroup.

As a specialization of the above abstract example of the inverse semigroup let us take $I = \mathfrak{M}_*^+$ and define $\mathcal{U}_{\omega, \omega'} := \mathcal{U}(\mathfrak{M})_{\omega}^{\omega'}$ for $\omega, \omega' \in \mathfrak{M}_*^+$. Here we consider $\mathcal{U}(\mathfrak{M})_{\omega}^{\omega'}$ as a set of partial isometries in Hilbert space $L^2 \Gamma(\mathbb{E}, \mathfrak{M}_*^+)$, see definition (2.58).

Example 3.2. Let $V_{ij} \subset \mathcal{U}_{ij}$ be such that $\bigcup_{i,j \in I} V_{ij} \cup \{0\}$ is closed with respect to (3.4) and (3.5). Then $\bigcup_{i,j \in I} V_{ij} \cup \{0\}$ is the inverse subsemigroup of $\mathcal{U}(\{p_i\}_{i \in I}) \cup \{0\}$.

Example 3.3. In the case when V_{ij} are one element subsets we will denote the inverse semigroup $\bigcup_{i,j \in I} V_{ij}$ by $\mathcal{V}(\{p_i\}_{i \in I})$. Note also that $\mathcal{V}(\{p_i\}_{i \in I})$ is the $(I \times I)$ -matrix unit in $p\mathfrak{M}p$, where $p = \sum_{i \in I} p_i$. For the definition of the $(I \times I)$ -matrix unit see e.g. [10], [19]. In the below we will call $\mathcal{V}(\{p_i\}_{i \in I})$ the **matrix unit inverse semigroup**.

The notion of the $(I \times I)$ -matrix unit plays an important role in studying of W^* -algebra structure for particular of I type W^* -algebras, e.g. see [10]. Recall that for any central projection z of a W^* -algebra \mathfrak{M} of type I one has decomposition $z = \sum_{i \in I} p_i$ given by a family $\{p_i\}_{i \in I}$ of mutually orthogonal, equivalent abelian projections. The cardinal number $\alpha := \sharp I$ does not depend on a choice of $\{p_i\}_{i \in I}$. Hence we conclude that for any α -homogeneous direct summand $z\mathfrak{M}$ of the I type W^* -algebra \mathfrak{M} one has abelian $(I \times I)$ -matrix unit inverse semigroup $\mathcal{V}(\{p_i\}_{i \in I})$. The inverse semigroups $\mathcal{V}(\{p_i\}_{i \in I})$ are isomorphic for different choices of the decompositions $z = \sum_{i \in I} p_i$.

Let $B_{part}(I)$ be the inverse semigroup of partial bijections of the set of indices I which appeared in the above examples. Let $\mathcal{J}(I) \subset B_{part}(I)$ be the subset of partial bijections $\varphi : A \rightarrow B$, where $A, B \subset I$, such that

$$U_{\varphi(i)i} \neq \emptyset. \quad (3.6)$$

for each $i \in A$.

We denote by $\mathcal{U}(I) \subset \mathcal{U}(\mathfrak{M})$ the subset of partial isometries defined by

$$u_{\varphi} := \sum_{i \in A} u_{\varphi(i)i}, \quad (3.7)$$

where $u_{\varphi(i)i} \in U_{\varphi(i)i}$.

Theorem 3.4. (i) The subset $\mathcal{U}(I) \subset \mathcal{U}(\mathfrak{M})$ is an inverse semigroup of partial isometries.

(ii) The subsemigroup E of idempotents of $\mathcal{U}(I)$ is a semilattice of orthogonal projections which are defined by

$$p_A := \sum_{i \in A} p_i, \quad A \subset I. \quad (3.8)$$

(iii) The map

$$\phi : \mathcal{U}(I) \ni u_\varphi \rightarrow \varphi \in \mathfrak{I}(I) \quad (3.9)$$

is a surjective morphism of the inverse semigroups.

Proof. (i) Taking another partial bijection $\psi : C \rightarrowtail D$ of $C, D \subset I$ and the corresponding partial isometry u_ψ we find that

$$u_\psi u_\varphi = \left(\sum_{j \in C} u_{\psi(j)j} \right) \left(\sum_{i \in A} u_{\varphi(i)i} \right) = \quad (3.10)$$

$$= \sum_{k \in \varphi^{-1}(B \cap C)} u_{(\psi \circ \varphi)(k)\varphi(k)} u_{\varphi(k)k},$$

$$u_\varphi^* = \sum_{i \in A} u_{\varphi(i)i}^* = \sum_{j \in B} u_{j\varphi^{-1}(j)}^*, \quad (3.11)$$

where

$$\psi \circ \varphi : \varphi^{-1}(B \cap C) \rightarrowtail \psi(B \cap C) \quad (3.12)$$

and

$$\varphi^{-1} : B \rightarrowtail A. \quad (3.13)$$

The product $u_{(\psi \circ \varphi)(k)\varphi(k)} u_{\varphi(k)k}$ belongs to $U_{(\psi \circ \varphi)(k)k}$ and $u_{\varphi(i)i}^* \in U_{\varphi^{-1}(j)j}$, where $j = \varphi(i)$. Thus $u_\psi u_\varphi \in \mathcal{U}(I)$ and $u_\varphi^* \in \mathcal{U}(I)$.

(ii) It follows from (3.10) and (3.11) that

$$u_\varphi^* u_\varphi = u_{\varphi^{-1}} u_\varphi = u_{id_A} = \sum_{i \in A} p_i$$

where $id_A : A \rightarrow A$ is identity partial bijection. Similarly we can show that $u_\varphi u_\varphi^* = \sum_{i \in B} p_i$.

(iii) From (3.10) and (3.11) follows that $\mathfrak{I}(I)$ is closed with respect to the superposition of partial bijections. The fact, that the map ϕ defined in (3.9) is a surjective morphism, follows from (3.10) and (3.11) and definition of $\mathfrak{I}(I)$. □

Remark 3.5. Taking partial isometries $u_\varphi \in \mathcal{U}(I)$ where φ is a bijection between one elements sets we find that the matrix unit inverse semigroup $\mathcal{U}(\{p_i\}_{i \in I})$ is an inverse subsemigroup of $\mathcal{U}(I)$.

In order to obtain an interesting examples of the inverse semigroup $\mathcal{V}(\{p_i\}_{i \in I})$, where $I = \{1, 2, \dots, N\}$ or $I = \mathbb{N}$, let us take a properly infinite W^* -algebra \mathfrak{M} . It follows from the Halving Theorem (see [10]) that there exists a family $\{p_i\}_{i \in I}$ of mutually orthogonal projections in \mathfrak{M} such that

$$p_i \sim \mathbf{1} \quad \text{and} \quad \sum_{i \in I} p_i = \mathbf{1}, \quad (3.14)$$

where $I = \{1, 2, \dots, N\}$ or $I = \mathbb{N} \cup \{\infty\}$. Since of $p_i \sim \mathbf{1}$ there are isometries $s_i \in \mathcal{U}(\mathfrak{M})$ satisfying $s_i^* s_i = \mathbf{1}$ and $s_i s_i^* = p_i$. One has $s_i s_j^* s_k s_l^* = \delta_{jk} s_i s_l^*$ and $(s_i s_j^*)^* = s_j s_i^*$. Hence $V_{ij} = \{s_i s_j^*\}$ is the self-adjoint system of the $I \times I$ matrix units, i.e. it is the inverse semigroup.

For $N > 1$ the $\mathcal{V}(\{p_i\}_{i \in I})$ is an inverse subsemigroup $\mathcal{V}_N \subset S_N$ of the Cuntz inverse semigroup S_N . The **Cuntz inverse semigroup** S_N consists of elements

$$s_{i_1} \dots s_{i_k} s_{j_1}^* \dots s_{j_l}^*$$

where $i_1, \dots, i_k, j_1, \dots, j_l \in I$, e.g. see [16]. Denoting k -tuples by $\alpha := (i_1, \dots, i_k)$ and the isometry $s_{i_1} \dots s_{i_k}$ by s_α we find that the product and the involution in Cuntz inverse semigroup S_N are given by

$$\begin{aligned} (s_\alpha s_{\beta\gamma}^*)(s_\beta s_\delta^*) &= s_\alpha s_{\gamma\delta}^* \\ (s_\alpha s_\beta^*)(s_{\beta\gamma} s_\delta^*) &= s_{\alpha\gamma} s_\delta^* \end{aligned} \quad (3.15)$$

and

$$(s_\alpha s_\beta^*)^* = s_\beta s_\alpha^* \quad (3.16)$$

respectively, see [17]. The inverse semigroup S_N generates the C^* -subalgebra of \mathfrak{M} isomorphic to Cuntz algebra \mathcal{O}_N [6]. Note that for $N = 1$ one obtains Toeplitz inverse semigroup.

Proposition 3.6. *W^* -algebra \mathfrak{M} is properly infinite if and only if it contains a Cuntz inverse semigroup S_N such that $\mathbf{1} \in S_N$, where $N \in \mathbb{N}$ or $N = \infty$.*

Proof. Summing up the facts presented above we find that any properly infinite W^* -algebra \mathfrak{M} contains a Cuntz inverse semigroup S_N and unit of \mathfrak{M} belongs to S_N .

Let us assume that $S_N \subset \mathfrak{M}$ and $\mathbf{1} \in S_N$. Then from (3.14) we have

$$z = \sum_{i \in I} p_i z \quad \text{and} \quad p_i z \sim z \quad (3.17)$$

for any central projection z of \mathfrak{M} . From (3.17) follows that any central projection z is equivalent to its not trivial subprojection. This means that \mathfrak{M} is properly infinite. \square

We recall that a **Clifford inverse semigroup** is an inverse semigroup \mathcal{S} whose idempotents are central, see [9].

Proposition 3.7. *Let $E \subset \mathcal{L}(\mathfrak{M})$ be a semilattice. We define*

$$\mathcal{S} := \bigcup_{p \in E} Z(p\mathfrak{M}p) \cap \mathcal{U}(\mathfrak{M}), \quad (3.18)$$

where $Z(p\mathfrak{M}p)$ is the center of the W^* -subalgebra $p\mathfrak{M}p \subset \mathfrak{M}$. Then \mathcal{S} is an inverse semigroup if $\mathcal{S} \subset \mathcal{U}(E)$. Additionally \mathcal{S} is a Clifford inverse semigroup.

Proof. Assuming $\mathcal{S} \subset \mathcal{U}(E)$ we obtain from (3.18) that for $u, v \in \mathcal{S}$ there are $p, q \in E$ such that $p = u^*u = uu^*$ and $q = v^*v = vv^*$. From

$$uv(uv)^*uv = uvv^*u^*uv = uu^*uvv^*v = uv$$

and from

$$uv = upqv = (pqupq)(pqvpq)$$

it follows that uv is a partial isometry and $uv \in pq\mathfrak{M}pq$. Since $u \in Z(p\mathfrak{M}p)$, $v \in Z(q\mathfrak{M}q)$ and $p\mathfrak{M}p \cap q\mathfrak{M}q = pq\mathfrak{M}pq$ we have $uv \in Z(pq\mathfrak{M}pq)$. The above shows that \mathcal{S} is an inverse semigroup. In order to prove $qu = uq$ for any $u \in \mathcal{S}$ and any $q \in E$ we observe that $u \in Z(p\mathfrak{M}p) \cap \mathcal{U}(\mathfrak{M})$ for some $p \in E$. Since $pq \in Z(p\mathfrak{M}p) \cap \mathcal{U}(\mathfrak{M})$ one obtains $qu = qpu = uqp = uq$. \square

Remark 3.8. The condition $\mathcal{S} \subset \mathcal{U}(E)$ from Proposition 3.18 is fulfilled, for example, if from $p \in E$ and $q \leq p$ follows $q \in E$.

Now we will describe the inverse semigroups $\mathcal{S} \subset \mathcal{U}(\mathfrak{M})$ generated by a single partial isometry $u \in \mathcal{U}(\mathfrak{M})$. According to [9] we will call such an inverse semigroup a **monogenic inverse semigroup**. By definition, see [8], the partial isometry $u \in \mathfrak{M}$ is a **power partial isometry** if u^k is a partial isometry for all $k \in \mathbb{N}$.

We conclude from Remark 3.1 that

$$up_{k+1} = p_k u \quad \text{and} \quad u^* p_k = p_{k+1} u^* \quad (3.19)$$

$$q_{l+1} u = u q_l \quad \text{and} \quad q_l u^* = u^* q_{l+1} \quad (3.20)$$

where

$$p_k := u^{*k} u^k, \quad q_l := u^l u^{*l}. \quad (3.21)$$

Lemma 3.9. *Projections p_k and q_l defined in (3.21) have the following properties:*

$$p_k p_l = p_{\max\{k, l\}} \quad (3.22)$$

$$q_k q_l = q_{\max\{k, l\}} \quad (3.23)$$

$$p_k q_l = q_l p_k. \quad (3.24)$$

Proof. Assuming $k > l$ we obtain

$$p_k = u^{*k}u^k = u^{*l}p_{k-l}u^l = p_l p_k \quad (3.25)$$

where the last equality in (3.25) follows from (3.19). Similarly we prove that $q_k = q_l q_k$.

Since $v^k u^l = u^{k+l}$ is a partial isometry the projections $p_k = u^{*k}u^k$ and $q_l = u^l u^{*l}$ commute, see Remark 3.1. This proves (3.24). \square

Theorem 3.10. *The partial isometry $u \in \mathcal{U}(\mathfrak{M})$ generates a monogenic inverse semigroup $\mathcal{S}_{\langle u \rangle}$ if and only if u is a power partial isometry. Every partial isometry from $\mathcal{S}_{\langle u \rangle}$ can be expressed in the form*

$$p_k q_l u^m \quad \text{or} \quad p_k q_l u^{*m} \quad (3.26)$$

where $k, l, m \in \mathbb{N}$, while $u^0 = 1$ and $u^{*0} = 1$. In particular the set of idempotents of $\mathcal{S}_{\langle u \rangle}$ consists of projections $p_k q_l$.

Proof. If $\mathcal{S}_{\langle u \rangle} \subset \mathcal{U}(\mathfrak{M})$ is a monogenic inverse semigroup then $u^k \in \mathcal{S}_{\langle u \rangle}$. So it is a partial isometry. So, u is power partial isometry.

Now let us assume that for any $k \in \mathbb{N}$ element u^k is a partial isometry. Then the relations (3.19), (3.20), (3.22), (3.23) and (3.24) are valid. Using these relations one transforms the arbitrary element

$$u^{k_1} u^{*l_1} u^{k_2} u^{*l_2} \dots u^{k_N} u^{*l_N} \quad (3.27)$$

to the product of three elements

$$p_k q_l u^m \quad \text{or} \quad p_k q_l u^{*m} \quad (3.28)$$

which satisfy

$$p_k q_l u^m (p_k q_l u^m)^* = p_k q_l q_m = p_k q_{\max\{l, m\}}$$

or

$$p_k q_l u^{*m} (p_k q_l u^{*m})^* = p_k q_l p_m = q_l p_{\max\{k, m\}}.$$

So, they are partial isometries. In such a way we show that the arbitrary element (3.27) generated by u is a partial isometry of the form (3.26) and it is an idempotent if and only if $m = 0$. \square

Corollary 3.11. *Each element of $\mathcal{S}_{\langle u \rangle}$ can be presented in the form*

$$u^k u^{*l} u^m \quad (3.29)$$

where $0 \leq k \leq l$, $0 \leq m \leq l$ and $l > 0$.

Proof. One obtains (3.29) from (3.26) using (3.19), (3.20), (3.22), (3.23) and (3.24). \square

Since an arbitrary monogenic inverse semigroup \mathcal{S} is isomorphic to some $\mathcal{S}_{\langle u \rangle}$ the statement of Corollary 3.11 is valid also for each monogenic inverse semigroup. We thus proved a statement known as the Gluskin theorem [7].

Example 3.4. If the generator u of $\mathcal{S}_{\langle u \rangle}$ is an isometry (co-isometry), i.e. $u^*u = 1$ ($uu^* = 1$), then the monogenic inverse semigroup $\mathcal{S}_{\langle u \rangle}$ is the Toeplitz inverse semigroup. In this case any element of $\mathcal{S}_{\langle u \rangle}$ can be written in the form $q_l u^m (p_k u^m)$.

Other examples of inverse semigroups are given in the following

Proposition 3.12. *A projection $p \in \mathfrak{M}$ and a unitary element $u \in \mathfrak{M}$ generate an inverse semigroup $\mathcal{S}_{\langle p, u \rangle} \subset \mathfrak{M}$ if and only if*

$$[p, u^k p u^{*k}] = 0 \quad (3.30)$$

for $k \in \mathbb{N}$.

Proof. Using for any $k \in \mathbb{N}$ the following notation

$$u^k := \begin{cases} u^k & \text{for } k \in \mathbb{N} \cup \{0\} \\ u^{*k} & \text{for } -k \in \mathbb{N} \end{cases}$$

we observe that any element $x \in \mathfrak{M}$ generated by the products of p and u has the following form

$$x = u^{l_1} p u^{l_2} p u^{l_3} \dots p u^{l_{N-1}} p u^{l_N}, \quad (3.31)$$

where $l_1, l_2, \dots, l_N \in \mathbb{Z}$. Let us assume (3.30). Then we have

$$[u^k p u^{-k}, u^l p u^{-l}] = 0 \quad (3.32)$$

for any $k, l \in \mathbb{Z}$. From (3.32) we obtain that $x^2 = x = x^*$ if and only if $l_1 + l_2 + \dots + l_N = 0$. Thus xx^* and x^*x are idempotents for any choice of $l_1, l_2, \dots, l_N \in \mathbb{Z}$. Hence elements of \mathfrak{M} given by (3.31) generate the inverse semigroup which we denote by $\mathcal{S}_{\langle p, u \rangle}$.

If $\mathcal{S}_{\langle p, u \rangle} \subset \mathfrak{M}$ is the inverse semigroup then the elements p and $u^k p u^{*k}$ commute since they are idempotents of $\mathcal{S}_{\langle p, u \rangle}$. \square

Corollary 3.13. *The orthogonal projections $p, 1-p \in \mathfrak{M}$ and a unitary element $u \in \mathfrak{M}$ generate an inverse semigroup $\mathcal{S}_{\langle p, 1-p, u \rangle}$ if and only if the condition (3.30) is fulfilled.*

Proof. The set $\mathcal{S}_{\langle 1-p, u \rangle}$ is an inverse semigroup if and only if p and u satisfy the condition (3.30). From this and from $[p, 1-p] = 0$ we have that $\mathcal{S}_{\langle p, 1-p, u \rangle}$ is the inverse semigroup if and only if (3.30) is fulfilled. \square

Finally let us give an example of inverse semigroup with a remarkable significance in quantum physics. To this end let us consider the sequence of bounded operators $a_1, a_2, \dots, a_N \in L^\infty(\mathcal{H})$, where $N \in \mathbb{N}$ or $N = \infty$, acting in the separable Hilbert space \mathcal{H} and satisfying the canonical anticommutation relations (CAR)

$$\begin{aligned} a_i a_j^* + a_j^* a_i &= \delta_{ij} 1 \\ a_i a_j + a_j a_i &= 0 \end{aligned} \quad (3.33)$$

It follows from (3.33) that the annihilation operators a_1, a_2, \dots, a_N and their adjoints $a_1^*, a_2^*, \dots, a_N^*$ (i.e. the creation operators) are partial isometries. The projections $P_i := a_i a_i^*$, $i = 1, 2, \dots, N$, and the complementary projections $Q_j := 1 - P_j = a_j^* a_j$, $j = 1, 2, \dots, n$ commute. So, they generate a semilattice $E \subset \mathcal{L}(\mathfrak{M})$.

Any element $x \in \mathfrak{M}$ generated by the products of the annihilation and creation operators has the following form

$$x = c P_\alpha Q_\beta a_\gamma^* a_\delta, \quad (3.34)$$

where $c \in \{-1, 1\}$ and the products $P_\alpha := P_{\alpha_1} P_{\alpha_2} \dots P_{\alpha_k}$, $Q_\beta := Q_{\beta_1} Q_{\beta_2} \dots Q_{\beta_l}$, $a_\gamma^* := a_{\gamma_1}^* a_{\gamma_2}^* \dots a_{\gamma_m}^*$, $a_\delta := a_{\delta_1} a_{\delta_2} \dots a_{\delta_n}$ are enumerated by the increasing sequences of indices from $\{1, 2, \dots, N\}$ such that for $A, B \in \{\alpha, \beta, \gamma, \delta\}$, $A \neq B$ implies that $A \cap B = \emptyset$.

The product of the two such elements is given by

$$xx' = c'' P_{\alpha''} Q_{\beta''} a_{\gamma''}^* a_{\delta''}, \quad (3.35)$$

where $c'' = cc'(-1)^{m-i+j-p+r}$ for $\gamma_i = \delta'_j$, $\delta_p = \gamma'_r$

$$\alpha'' = \alpha \cup \alpha' \cup (\delta \cap \gamma') \setminus (\delta' \cup \gamma)$$

$$\beta'' = \beta \cup \beta' \cup (\delta' \cap \gamma) \setminus (\delta \cup \gamma')$$

$$\gamma'' = \gamma \cup \gamma' \setminus (\delta \cup \delta')$$

$$\delta'' = \delta \cup \delta' \setminus (\gamma \cup \gamma')$$

if $\alpha \cap \gamma' = \emptyset$, $\beta \cap \delta' = \emptyset$, $\alpha \cap \beta' = \emptyset$, $\beta \cap \alpha' = \emptyset$, $\gamma \cap \gamma' = \emptyset$, $\delta \cap \delta' = \emptyset$. In the opposite case one has $xx' = 0$.

In particular the products of the conjugated elements take the following forms:

$$xx^* = c P_\alpha Q_\beta a_\gamma^* a_\delta a_\gamma Q_\beta P_\alpha c = P_{\alpha \cup \beta} Q_{\beta \cup \gamma} \quad (3.36)$$

$$x^* x = a_\delta^* a_\gamma Q_\beta P_\alpha P_\alpha Q_\beta a_\gamma^* a_\delta = P_{\alpha \cup \gamma} Q_{\beta \cup \delta}$$

Concluding, we see that subset $\mathcal{S} \subset L^\infty(\mathcal{H})$ consisting of elements (3.34) is an inverse semigroup. Let us call it the CAR inverse semigroup.

Ending this section we go back to the Proposition 2.10 and describe a faithful representation of the inverse semigroup of local bisections $\mathfrak{B}_{loc}(\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+)$ of the action groupoid $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ in the W^* -algebra $\overline{\rho}(\mathfrak{M})'$. To this end let us note the local bisection $\sigma \in \mathfrak{B}_{loc}(\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+)$ is a map

$$\sigma_\Omega : \Omega \ni \omega \mapsto \sigma_\Omega(\omega) = (\tilde{\sigma}(\omega), \omega) \in \mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+ \quad (3.37)$$

of $\Omega \subset \mathfrak{M}_*^+$ which satisfies

$$\mathbf{s}(\sigma(\omega)) = \omega,$$

$$\mathbf{t}(\sigma(\omega)) = I_{*\tilde{\sigma}(\omega)} \omega.$$

Using groupoids monomorphism (2.60) we define the representation

$$\phi : \mathfrak{B}_{loc}(\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+) \rightarrow \overline{\rho}(\mathfrak{M})' \quad (3.38)$$

of $\mathfrak{B}_{loc}(\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+)$ in $\overline{\rho}(\mathfrak{M})'$ as follows

$$\phi(\sigma_\Omega) := \sum_{\omega \in \Omega} (\iota \circ \phi)(\tilde{\sigma}(\omega), \omega). \quad (3.39)$$

The idempotent corresponding to $\phi(\sigma_\Omega)$ is given by

$$\phi(\sigma_\Omega)^* \phi(\sigma_\Omega) = \sum_{\omega \in \Omega} id_\omega$$

where id_ω is the identity in $\overline{\mathbb{E}}_\omega \subset L^2\Gamma(\mathbb{E}, \mathfrak{M}_*^+)$.

4 Topologies and Banach manifold structure of groupoids $\mathcal{G}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M})$

There are the following locally convex topologies considered on the W^* -algebra \mathfrak{M} : the uniform topology, the Arens-Mackey topology $\tau(\mathfrak{M}, \mathfrak{M}_*)$, the strong $*$ -topology $s^*(\mathfrak{M}, \mathfrak{M}_*)$, the strong topology $s(\mathfrak{M}, \mathfrak{M}_*)$, the σ -weak topology $\sigma(\mathfrak{M}, \mathfrak{M}_*)$, see e. g. [18]. All these topologies define corresponding topologies on the groupoids $\mathcal{G}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M})$. Hence, the natural question arises for which of the topologies listed above the groupoids are topological groupoids.

Let us start from $\mathcal{G}(\mathfrak{M})$.

Proposition 4.1. *For a infinite-dimentional W^* -algebra \mathfrak{M} the groupoid $\mathcal{G}(\mathfrak{M})$ is not a topological groupoid with respect to any topology of \mathfrak{M} mentioned above.*

Proof. Let us take $p \in \mathcal{L}(\mathfrak{M})$ and define $x_n \in \mathcal{G}(\mathfrak{M})$ by

$$x_n = p + \frac{1}{n}(1 - p), \quad n \in \mathbb{N}.$$

One has

$$\mathbf{s}(x_n) = \mathbf{t}(x_n) = 1 \quad \text{and} \quad \mathbf{s}(p) = \mathbf{t}(p) = p.$$

Since the uniform limit of x_n is

$$p = \lim_{n \rightarrow \infty} x_n,$$

we see that source and target maps of $\mathcal{G}(\mathfrak{M})$ are not continuous. Thus we obtain that $\mathcal{G}(\mathfrak{M})$ is not a topological one. Note that the above consideration does not depend on the choice of topology on \mathfrak{M} . \square

The case of the groupoid $\mathcal{U}(\mathfrak{M})$ is much better than that of $\mathcal{G}(\mathfrak{M})$. Let us begin our considerations from the uniform topology. Since all algebraic operations in \mathfrak{M} and the $*$ -involution are uniformly continuous and groupoid maps are expressed by these operations we conclude that the groupoid $\mathcal{U}(\mathfrak{M})$ is a topological groupoid with respect to the uniform topology. Let us remark also

that $\mathcal{U}(\mathfrak{M})$ is uniformly closed in \mathfrak{M} and $\mathcal{L}(\mathfrak{M})$ is uniformly closed in $\mathcal{U}(\mathfrak{M})$. Note also that the set $\mathcal{U}(\mathfrak{M})^{(2)} = (\mathfrak{s} \times \mathfrak{t})^{-1}(\{(p, p) : p \in \mathcal{L}(\mathfrak{M})\})$ is closed in $\mathcal{U}(\mathfrak{M}) \times \mathcal{U}(\mathfrak{M})$.

The groupoid of partial isometries $\mathcal{U}(\mathfrak{M})$ is not topological with respect to the $\sigma(\mathfrak{M}, \mathfrak{M}_*)$ -topology (the weak *-topology) and with respect to the $s(\mathfrak{M}, \mathfrak{M}_*)$ -topology (the strong topology). The reason is that the product map (2.6) is not continuous with respect to $\sigma(\mathfrak{M}, \mathfrak{M}_*)$ -topology and the involution (2.7) is not continuous with respect to the $s(\mathfrak{M}, \mathfrak{M}_*)$ -topology.

The Arens-Mackey topology $\tau(\mathfrak{M}, \mathfrak{M}_*)$ coincides with the s^* -strong topology $s^*(\mathfrak{M}, \mathfrak{M}_*)$ on the bounded parts of \mathfrak{M} , see [18]. So both of them induce on $\mathcal{U}(\mathfrak{M})$ the same topology. So, without loss of generality we can restrict our consideration to the $s^*(\mathcal{U}(\mathfrak{M}))$ -topology of $\mathcal{U}(\mathfrak{M})$.

Let us take the closed unit ball $\mathcal{B} = \{x \in \mathfrak{M} : \|x\| \leq 1\}$ in \mathfrak{M} . The product map $\mathcal{B} \times \mathcal{B} \ni (x, y) \mapsto xy \in \mathcal{B}$ restricted to \mathcal{B} as well as the *-involution are continuous with respect to $s^*(\mathcal{B}, \mathfrak{M}_*)$ -topology. From the above we conclude:

Proposition 4.2. *The groupoid $\mathcal{U}(\mathfrak{M})$ of partial isometries is a topological groupoid with respect to the $s^*(\mathcal{U}(\mathfrak{M}), \mathfrak{M}_*)$ -topology.*

Let us define on $\mathfrak{M}_* \cong \{(p, \omega) \in \mathcal{L}(\mathfrak{M}) \times \mathfrak{M}_* : p = r_*(\omega)\}$ (respectively $\mathfrak{M}_* \cong \{(p, \omega) \in \mathcal{L}(\mathfrak{M}) \times \mathfrak{M}_* : p = l_*(\omega)\}$) the topology $\mathfrak{T}_{\mathfrak{M}_*}$ as the topology inherited from the product topology of $\mathcal{L}(\mathfrak{M}) \times \mathfrak{M}_*$. The moment map $r_* : \mathfrak{M}_* \rightarrow \mathcal{L}(\mathfrak{M})$ (respectively $l_* : \mathfrak{M}_* \rightarrow \mathcal{L}(\mathfrak{M})$) is continuous with respect to $\mathfrak{T}_{\mathfrak{M}_*}$. Since the topology $\mathfrak{T}_{\mathfrak{M}_*}$ of \mathfrak{M}_* is stronger than the uniform topology of \mathfrak{M}_* the action (2.34) (respectively (2.35)) is also continuous with respect to $\mathfrak{T}_{\mathfrak{M}_*}$.

Let us define the set

$$\mathcal{P} := \{\omega \in \mathfrak{M}_* : l_*(\omega) = r_*(\omega)\}. \quad (4.1)$$

We conclude from the Proposition (2.8) that subsets $\mathfrak{M}_*^+ \subset \mathfrak{M}_*^h \subset \mathcal{P}(\mathfrak{M}_*) \subset \mathfrak{M}$ of positive normal functionals, selfadjoint functionals and $\mathcal{P}(\mathfrak{M})$ are invariant with respect to the predual inner action $I_* : \mathcal{U}(\mathfrak{M}) \times \mathfrak{M}_* \rightarrow \mathfrak{M}_*$. The groupoid $\mathcal{U}(\mathfrak{M})$ acts on \mathfrak{M}_*^h , \mathfrak{M}_*^+ and $\mathcal{P}(\mathfrak{M})$ the actions are continuous with respect to their $\mathfrak{T}_{\mathfrak{M}_*}$ -topology. Since

$$s_*(I_{*u}\omega) = s_*(u\omega u^*) = uu^* = \mathfrak{t}(u)$$

$$I_{*u}(I_{*v}\omega) = I_{*u}(v\omega v^*) = uv\omega v^*u^* = uv\omega(uv)^* = I_{*uv}\omega$$

$$I_{*E(s_*(\omega))}\omega = I_{*s_*(\omega)}\omega = u^*u\omega u^*u = \omega$$

we see that the groupoid $\mathcal{U}(\mathfrak{M})$ acts on $\mathcal{P}(\mathfrak{M}_*)$ in the continuous way with respect to $\mathfrak{T}_{\mathfrak{M}_*}$ topology of $\mathcal{P}(\mathfrak{M}_*)$.

Summarizing the above considerations and applying the construction presented in the Appendix we have the following:

Theorem 4.3. *(i) The groupoids $\mathcal{U}(\mathfrak{M}) *_l \mathfrak{M}$, $\mathcal{U}(\mathfrak{M}) *_r \mathfrak{M}$, $\mathcal{U}(\mathfrak{M}) *_J \mathfrak{M}$, $\mathcal{U}(\mathfrak{M}) *_\mathfrak{M}^h$ and $\mathcal{U}(\mathfrak{M}) *_\mathfrak{M}^+$ are topological groupoids with respect to the relative topology inherited from the product uniform topology of $\mathcal{U}(\mathfrak{M}) \times \mathfrak{M}$.*

- (ii) The groupoids $\mathcal{U}(\mathfrak{M}) *_{l_*} \mathfrak{M}_*$, $\mathcal{U}(\mathfrak{M}) *_{r_*} \mathfrak{M}_*$, $\mathcal{U}(\mathfrak{M}) * \mathcal{P}(\mathfrak{M}_*)$, $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ and $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^h$ are topological groupoids with respect to the relative topology inherited from the product uniform topology of $\mathcal{U}(\mathfrak{M}) \times \mathfrak{M}_*$.
- (iii) The groupoids listed above cover the groupoid $\mathcal{U}(\mathfrak{M})$.

Now we show that the groupoids $\mathcal{G}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M})$ have canonically defined structure of complex and real Banach manifold respectively. Let us begin from definition of complex Banach manifold structure on the lattice $\mathcal{L}(\mathfrak{M})$ of projections of W^* -algebra \mathfrak{M} . For this reason for any $p \in \mathcal{L}(\mathfrak{M})$ by $\Pi_p \subset \mathcal{L}(\mathfrak{M})$ we denote the subset of projections $q \in \mathcal{L}(\mathfrak{M})$ such that

$$q \wedge (1 - p) = 0 \quad \text{and} \quad q \vee (1 - p) = 1, \quad (4.2)$$

where " \wedge " and " \vee " are joint- and meet-operations on the projections in the lattice $\mathcal{L}(\mathfrak{M})$. Since for any pair of projections $e, f \in \mathcal{L}(\mathfrak{M})$ one has

$$(e \vee f) - e \sim f - (e \wedge f),$$

see [19], taking $e = 1 - p$ and $f = q$ we obtain that $q \sim p$. So we have $\Pi_p \subset \mathcal{O}_p$ and thus $\Pi_{p'} \cap \Pi_p = \emptyset$ if $p' \not\sim p$. The inverse statement, i.e. that $p' \sim p$ implies $\Pi_{p'} \cap \Pi_p \neq \emptyset$ is not true in general case. For example for infinite W^* -algebra we can take $p \neq 1$ such that $p \sim 1$. Then $\Pi_1 = \{1\}$ and thus $\Pi_1 \cap \Pi_p = \emptyset$. Condition (4.2) is equivalent to existence of Banach splitting

$$\mathfrak{M} = q\mathfrak{M} \oplus (1 - p)\mathfrak{M} \quad (4.3)$$

of \mathfrak{M} on the right W^* -ideals. Using (4.3) we decompose

$$p = x - y \quad (4.4)$$

the projection p on two elements $x \in q\mathfrak{M}p$ and $y \in (1 - p)\mathfrak{M}p$. In such a way we define the map $\varphi_p : \Pi_p \xrightarrow{\sim} (1 - p)\mathfrak{M}p$ by

$$\varphi_p(q) := y. \quad (4.5)$$

Let us show that φ_p is a bijection of Π_p on the Banach space $(1 - p)\mathfrak{M}p$. To this end for any $y \in (1 - p)\mathfrak{M}p$ we define x by equality (4.4) and note that

$$p = px, \quad xp = x \quad \text{and} \quad x^2 = x. \quad (4.6)$$

Thus the left multiplication maps L_p and L_x on \mathfrak{M} satisfy

$$L_p = L_p \circ L_x, \quad L_x = L_x \circ L_p \quad \text{and} \quad L_x \circ L_x = L_x \quad (4.7)$$

and

$$\text{Ker } L_x = (1 - x)\mathfrak{M} = (1 - p)\mathfrak{M} = \text{Ker } L_p. \quad (4.8)$$

From (4.8) we have

$$\mathfrak{M} = x\mathfrak{M} \oplus (1 - p)\mathfrak{M}, \quad (4.9)$$

where $x\mathfrak{M}$ is right ideal of W^* -algebra generated by $x \in \mathfrak{M}$. Let us also note that

$$L_x : p\mathfrak{M} \xrightarrow{\sim} x\mathfrak{M} \quad \text{and} \quad L_p : x\mathfrak{M} \xrightarrow{\sim} p\mathfrak{M} \quad (4.10)$$

are mutually inverse isomorphisms of the corresponding right W^* -ideals.

The left support $l(x)$ of $x \in \mathfrak{M}$ is the identity in W^* -subalgebra $x\mathfrak{M} \cap (x\mathfrak{M})^*$. Thus $l(x) \in x\mathfrak{M}$. This shows that $x\mathfrak{M} = l(x)\mathfrak{M}$ and

$$\mathfrak{M} = l(x)\mathfrak{M} \oplus (1-p)\mathfrak{M}, \quad (4.11)$$

i.e. $l(x) \in \Pi_p$. In such a way we prove that φ_p has the inverse defined by

$$\varphi_p^{-1}(y) := l(p+y). \quad (4.12)$$

Proposition 4.4. *If $x \in q\mathfrak{M}p$ is defined by the decomposition (4.4) then $x \in \mathcal{G}(\mathfrak{M})$ and $\mathbf{s}(x) = p$ and $\mathbf{t}(x) = q$. So one has section $\sigma_p : \Pi_p \rightarrow \mathbf{t}^{-1}(\Pi_p) \subset \mathcal{G}(\mathfrak{M})$ defined by*

$$\sigma_p(q) := x. \quad (4.13)$$

Proof. The above follows from (4.10) and from $x\mathfrak{M} = l(x)\mathfrak{M} = q\mathfrak{M}$. \square

We conclude from the above that maps $\varphi_p : \Pi_p \rightarrow (1-p)\mathfrak{M}p$, where $p \in \mathcal{L}(\mathfrak{M})$, define a canonical atlas on $\mathcal{L}(\mathfrak{M})$. Recall that $\bigcup_{p \in \mathcal{L}(\mathfrak{M})} \Pi_p = \mathcal{L}(\mathfrak{M})$. This atlas is modeled by the family of Banach spaces $(1-p)\mathfrak{M}p$. If projections q and p are equivalent, i.e. if there exists $x \in \mathcal{G}(\mathfrak{M})$ such that $p = \mathbf{s}(x)$ and $p' = \mathbf{t}(x)$, then the Banach spaces $(1-p)\mathfrak{M}p$ and $(1-p')\mathfrak{M}p'$ are isomorphic.

Now we find the explicite formulae for the transitions maps

$$\varphi_p \circ \varphi_{p'}^{-1} : \varphi_{p'}(\Pi_p \cap \Pi_{p'}) \rightarrow \varphi_p(\Pi_p \cap \Pi_{p'}) \quad (4.14)$$

in the case when $\Pi_p \cap \Pi_{p'} \neq \emptyset$. For this reason let us take for $q \in \Pi_p \cap \Pi_{p'}$ the following splittings

$$\begin{aligned} \mathfrak{M} &= q\mathfrak{M} \oplus (1-p)\mathfrak{M} = p\mathfrak{M} \oplus (1-p)\mathfrak{M} \\ \mathfrak{M} &= q\mathfrak{M} \oplus (1-p')\mathfrak{M} = p'\mathfrak{M} \oplus (1-p')\mathfrak{M}. \end{aligned} \quad (4.15)$$

The splittings (4.15) lead to the corresponding decompositions of p and p'

$$\begin{aligned} p &= x - y & p &= a + b \\ p' &= x' - y' & 1 - p &= c + d \end{aligned} \quad (4.16)$$

where $x \in q\mathfrak{M}p$, $y \in (1-p)\mathfrak{M}p$, $x' \in q\mathfrak{M}p'$, $y' \in (1-p')\mathfrak{M}p'$, $a \in p'\mathfrak{M}p$, $b \in (1-p')\mathfrak{M}p$, $c \in p'\mathfrak{M}(1-p)$ and $d \in (1-p')\mathfrak{M}(1-p)$. Combining equations from (4.16) we obtain

$$q = \iota(x') + y'\iota(x') \quad (4.17)$$

$$q = (a + cy)\iota(x) + (b + dy)\iota(x). \quad (4.18)$$

Comparing (4.17) and (4.18) we find that

$$\iota(x') = (a + cy)\iota(x) \quad (4.19)$$

$$y'\iota(x') = (b + dy)\iota(x). \quad (4.20)$$

After substitution (4.19) into (4.20) and noting that $\mathbf{t}(a + cy) \leq p'$ we get

$$y' = (\varphi_{p'} \circ \varphi_p^{-1})(y) = (b + dy)\iota(a + cy). \quad (4.21)$$

All operations involved in the right-hand-side of equality (4.21) are smooth. Thus we conclude that the atlas (Π_p, φ_p) , $p \in \mathcal{L}(\mathfrak{M})$ defines on $\mathcal{L}(\mathfrak{M})$ the structure of a complex Banach manifold of type \mathfrak{G} , see [3], where \mathfrak{G} is the set of Banach spaces $(1 - p)\mathfrak{M}p$ indexed by elements $p \in \mathcal{L}(\mathfrak{M})$. See also [1] for the investigation of the infinite-dimensional Grassmannian as a homogeneous spaces of the Banach-Lie group $U(\mathfrak{M})$. Note, that when \mathfrak{M} is finite W^* -algebra then the orbits of the inner action of the groupoid $\mathcal{U}(\mathfrak{M})$ and the orbits of the inner action of unitary group $U(\mathfrak{M})$ on the lattice $\mathcal{L}(\mathfrak{M})$ coincide.

Now let us introduce a structure of the Banach smooth manifold on the groupoid $\mathcal{G}(\mathfrak{M})$.

For this reason taking $p, \tilde{p} \in \mathcal{L}(\mathfrak{M})$ we define the covering of $\mathcal{G}(\mathfrak{M})$ by subsets:

$$\Omega_{\tilde{p}p} := \mathbf{t}^{-1}(\Pi_{\tilde{p}}) \cap \mathbf{s}^{-1}(\Pi_p). \quad (4.22)$$

Let us note here that $\Omega_{\tilde{p}p} \neq \emptyset$ if and only if $\tilde{p} \sim p$. Note also that the set Ω_{pp} is a subgroupoid of $\mathcal{G}(\mathfrak{M})$. If $\Omega_{\tilde{p}p} \neq \emptyset$ then one has the one-to-one map

$$\psi_{\tilde{p}p} : \Omega_{\tilde{p}p} \rightarrow (1 - \tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1 - p)\mathfrak{M}p \quad (4.23)$$

of $\Omega_{\tilde{p}p}$ on an open subset of the direct sum of the Banach subspaces of the W^* -algebra \mathfrak{M} . This map we define by

$$\psi_{\tilde{p}p}(x) := (\varphi_{\tilde{p}}(\mathbf{t}(x)), \iota(\sigma_{\tilde{p}}(\mathbf{t}(x)))x\sigma_p(\mathbf{s}(x)), \varphi_p(\mathbf{s}(x))), \quad (4.24)$$

where $\sigma_p(q) \in q\mathfrak{M}p$ and $\varphi_p(q) \in (1 - p)\mathfrak{M}p$ are obtained from the decomposition

$$p = \sigma_p(q) + \varphi_p(q) \quad (4.25)$$

of p with respect to (4.3). Recall that $\sigma_p : \pi_p \rightarrow \mathbf{t}^{-1}(\Pi_p)$ is a section defined in (4.13).

The map $\psi_{\tilde{p}p}^{-1} : \psi_{\tilde{p}p}(\Omega_{\tilde{p}p}) \rightarrow \Omega_{\tilde{p}p}$ inverse to (4.24) looks as follows

$$\psi_{\tilde{p}p}^{-1}(\tilde{y}, z, y) := \sigma_{\tilde{p}}(\tilde{q})z\iota(\sigma_p(q)) = (\tilde{p} + \tilde{y})z\iota(p + y) \quad (4.26)$$

where $\tilde{q} = l(\tilde{p} + \tilde{y})$ and $q = l(p + y)$ are left supports of $\tilde{p} + \tilde{y}$ and $p + y$ respectively. The transition maps

$$\psi_{\tilde{p}'p'} \circ \psi_{\tilde{p}p}^{-1} : \psi_{\tilde{p}p}(\Omega_{\tilde{p}'p'} \cap \Omega_{\tilde{p}p}) \rightarrow \psi_{\tilde{p}'p'}(\Omega_{\tilde{p}'p'} \cap \Omega_{\tilde{p}p})$$

for $(\tilde{y}, z, y) \in \psi_{\tilde{p}p}(\Omega_{\tilde{p}'p'} \cap \Omega_{\tilde{p}p})$ are given by

$$(\psi_{\tilde{p}'p'} \circ \psi_{\tilde{p}p}^{-1})(\tilde{y}, z, y) := (\tilde{y}', z', y'), \quad (4.27)$$

where

$$\tilde{y}' = (\varphi_{\tilde{p}'} \circ \varphi_{\tilde{p}}^{-1})(\tilde{y}) = (\tilde{b} + \tilde{d}\tilde{y})\iota(\tilde{a} + \tilde{c}\tilde{y}) \quad (4.28)$$

$$y' = (\varphi_{p'} \circ \varphi_p^{-1})(y) = (b + dy)\iota(a + cy) \quad (4.29)$$

and

$$z' = \iota(\tilde{p}' + \tilde{y}')(\tilde{p} + \tilde{y})z\iota(p + y)(p' + y'). \quad (4.30)$$

We note that all maps in (4.28), (4.29), (4.30) are smooth. Thus we conclude that

$$(\Omega_{\tilde{p}p}, \psi_{\tilde{p}p} : \Omega_{\tilde{p}p} \rightarrow (1 - \tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1 - p)\mathfrak{M}p), \quad (4.31)$$

where $(p, \tilde{p}) \in \mathcal{L}(\mathfrak{M}) \times \mathcal{L}(\mathfrak{M})$ are pairs of equivalent projections, form smooth atlas on the groupoid $\mathcal{G}(\mathfrak{M})$ in sense of [3]. The smooth (analytic) Banach manifold structure of $\mathcal{G}(\mathfrak{M})$ has type \mathfrak{G} , where \mathfrak{G} is the set of Banach spaces $(1 - \tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1 - p)\mathfrak{M}p$ indexed by the pair of equivalent elements of $\mathcal{L}(\mathfrak{M})$.

Theorem 4.5. *The groupoid $\mathcal{G}(\mathfrak{M})$ is a Banach-Lie groupoid on base $\mathcal{L}(\mathfrak{M})$ with respect to the smooth (analytic) Banach manifold structure of type \mathfrak{G} defined by the atlas (4.31).*

Proof. We show that all groupoids maps and the groupoid product are smooth (analytic) with respect to the considered Banach manifold structure.

(i) For source and target map we have

$$(\varphi_p \circ \mathbf{s} \circ \psi_{\tilde{p}p}^{-1})(\tilde{y}, z, y) = y, \quad (4.32)$$

$$(\varphi_{\tilde{p}} \circ \mathbf{t} \circ \psi_{\tilde{p}p}^{-1})(\tilde{y}, z, y) = \tilde{y}. \quad (4.33)$$

We assumed in (4.32) and (4.33) that $(\tilde{y}, z, y) \in \psi_{\tilde{p}p}(\Omega_{\tilde{p}p})$, $\mathbf{s}(\psi_{\tilde{p}p}^{-1}(\tilde{y}, z, y)) \in \Pi_p$ and $\mathbf{t}(\psi_{\tilde{p}p}^{-1}(\tilde{y}, z, y)) \in \Pi_{\tilde{p}}$ respectively. We conclude from (4.32) and (4.33) that $\varphi_p \circ \mathbf{s} \circ \psi_{\tilde{p}p}^{-1}$ and $\varphi_{\tilde{p}} \circ \mathbf{t} \circ \psi_{\tilde{p}p}^{-1}$ are smooth (analytic) submersions.

(ii) For identity section $\varepsilon : \mathcal{L}(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M})$ we have

$$(\psi_{\tilde{p}p} \circ \varepsilon \circ \varphi_p^{-1})(y) = (\varphi_{\tilde{p}} \circ \varphi_p^{-1}(y), \iota(\sigma_{\tilde{p}}(\varphi_p^{-1}(y)))\sigma_p(\varphi_p^{-1}(y)), y), \quad (4.34)$$

where $y \in \varphi_p(\Pi_p)$. Since $\sigma_{\tilde{p}} : \Pi_{\tilde{p}} \rightarrow \mathbf{t}^{-1}(\Pi_{\tilde{p}})$ and $\sigma_p : \Pi_p \rightarrow \mathbf{t}^{-1}(\Pi_p)$ are smooth (analytic) sections we obtain that $\psi_{\tilde{p}p} \circ \varepsilon \circ \varphi_p^{-1}$ is smooth (analytic) map too.

(iii) The inverse map $\iota : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M})$ takes $\Omega_{\tilde{p}p}$ onto $\Omega_{p\tilde{p}}$ and we have

$$(\psi_{p\tilde{p}} \circ \iota \circ \psi_{\tilde{p}p}^{-1})(\tilde{y}, z, y) = (y, \iota(z), \tilde{y}). \quad (4.35)$$

Thus ι is a complex smooth (analytic) map.

- (iv) Let us take $x_1 \in \Omega_{\tilde{p}_1 p_1}$ and $x_2 \in \Omega_{\tilde{p}_2 p_2}$ such that $\mathbf{s}(x_1) = \mathbf{t}(x_2) \in \Pi_{\tilde{p}_2} \cap \Pi_{p_1}$. Assuming $\psi_{\tilde{p}_1 p_1}(x_1) = (\tilde{y}_1, z_1, y_1)$ and $\psi_{\tilde{p}_2 p_2}(x_2) = (\tilde{y}_2, z_2, y_2)$ we obtain that

$$\begin{aligned} \psi_{\tilde{p}_1 p_2}(\psi_{\tilde{p}_1 p_1}^{-1}(\tilde{y}_1, z_1, y_1) \psi_{\tilde{p}_2 p_2}^{-1}(\tilde{y}_2, z_2, y_2)) = \\ = (\tilde{y}_1, z_1 \iota(\sigma_{p_1}(\varphi_{p_1}^{-1}(y_1))) (\sigma_{\tilde{p}_2}(\varphi_{\tilde{p}_2}^{-1}(\tilde{y}_2))) z_2, y_2). \end{aligned} \quad (4.36)$$

Summing up we conclude that $\mathcal{G}(\mathfrak{M})$ is a Banach-Lie groupoid. \square

In order to investigate the structure of real Banach manifold on $\mathcal{U}(\mathfrak{M})$ we recall that one can define $\mathcal{U}(\mathfrak{M})$ as the set of the fixed points of the groupoid automorphism $J : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M})$, see (2.10). Next expressing $J : \Omega_{\tilde{p}p} \rightarrow \Omega_{\tilde{p}p}$ in the coordinates

$$\Omega_{\tilde{p}p} \ni x \mapsto \psi_{\tilde{p}p}(x) = (\tilde{y}, z, y) \in (1 - \tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1 - p)\mathfrak{M}p$$

we find that

$$\begin{aligned} \left(\psi_{\tilde{p}p} \circ J \circ \psi_{\tilde{p}p}^{-1} \right) (\tilde{y}, z, y) = \\ = \left(\tilde{y}, \iota(\sigma_{\tilde{p}}(\varphi_{\tilde{p}}^{-1}(\tilde{y}))^* \sigma_{\tilde{p}}(\varphi_{\tilde{p}}^{-1}(\tilde{y}))) \iota(z^*) \sigma_p(\varphi_p^{-1}(y))^* \sigma_p(\varphi_p^{-1}(y)), y \right), \end{aligned} \quad (4.37)$$

where $z \in \mathcal{G}(\mathfrak{M})_{\tilde{p}}^{\tilde{p}} \subset \tilde{p}\mathfrak{M}p$. Note that $\mathcal{G}(\mathfrak{M})_{\tilde{p}}^{\tilde{p}}$ is an open subset of the Banach subspaces $\tilde{p}\mathfrak{M}p$. Since $J^2(x) = x$ for $x \in \mathcal{U}(\mathfrak{M})$ one has

$$(DJ(x))^2 = \mathbf{1} \quad (4.38)$$

for $DJ(x) : T_x \mathcal{G}(\mathfrak{M}) \rightarrow T_x \mathcal{G}(\mathfrak{M})$. Thus one obtains a splitting of the tangent space

$$T_x \mathcal{G}(\mathfrak{M}) = T_x^+ \mathcal{G}(\mathfrak{M}) \oplus T_x^- \mathcal{G}(\mathfrak{M}) \quad (4.39)$$

defined by the Banach space projections

$$P_{\pm}(x) := \frac{1}{2} (\mathbf{1} \pm DJ(x)). \quad (4.40)$$

The Frechét derivative $D\iota(z)$ of the inversion map

$$\iota : \mathcal{G}(\mathfrak{M})_{\tilde{p}}^{\tilde{p}} \ni z \mapsto \iota(z) \in \mathcal{G}(\mathfrak{M})_{\tilde{p}}^{\tilde{p}}$$

at the point z is given by

$$D\iota(z) \Delta z = -\iota(z) \Delta z \iota(z), \quad (4.41)$$

where $\Delta z \in \tilde{p}\mathfrak{M}p$. Thus for $\Delta \tilde{y} \in (1 - \tilde{p})\mathfrak{M}\tilde{p}$, $\Delta z \in \tilde{p}\mathfrak{M}p$, $\Delta y \in (1 - p)\mathfrak{M}p$ we obtain

$$D \left(\psi_{\tilde{p}p} \circ J \circ \psi_{\tilde{p}p}^{-1} \right) (\tilde{y}, z, y) (\Delta \tilde{y}, \Delta z, \Delta y) = \quad (4.42)$$

$$= (\Delta \tilde{y}, -\tilde{g}^{-2}(\tilde{y})D\tilde{g}^2(\tilde{y}) \Delta \tilde{y}\tilde{g}^{-2}(\tilde{y})\iota(z^*)g^2(y) - \tilde{g}^{-2}(\tilde{y})\iota(z^*) \Delta z^*\iota(z^*)g^2(y) + \\ + \tilde{g}^{-2}(\tilde{y})\iota(z^*)Dg^2(y) \Delta y, \Delta y),$$

where

$$\tilde{g}(\tilde{y}) := [\sigma_{\tilde{p}}(\varphi_{\tilde{p}}^{-1}(\tilde{y}))^* \sigma_{\tilde{p}}(\varphi_{\tilde{p}}^{-1}(\tilde{y}))]^{\frac{1}{2}} \quad \text{and} \quad g(y) := [\sigma_p(\varphi_p^{-1}(y))^* \sigma_p(\varphi_p^{-1}(y))]^{\frac{1}{2}}$$

are elements of $G(\tilde{p}\mathfrak{M}\tilde{p})$ and $G(p\mathfrak{M}p)$ respectively.

If $x \in \Omega_{\tilde{p}p} \cap \mathcal{U}(\mathfrak{M})$ then from (4.37) one has

$$\tilde{g}^2(\tilde{y})zg^{-2}(y) = \iota(z^*). \quad (4.43)$$

It follows from (4.42) that $\Delta z \in T_x^+ \mathcal{G}(\mathfrak{M})$ if and only if it satisfies

$$\Delta z = -\tilde{g}^{-2}(\tilde{y})D\tilde{g}^2(\tilde{y}) \Delta \tilde{y}\tilde{g}^{-2}(\tilde{y})\iota(z^*)g^2(y) + \\ -\tilde{g}^{-2}(\tilde{y})\iota(z^*) \Delta z^*\iota(z^*)g^2(y) + \tilde{g}^{-2}(\tilde{y})\iota(z^*)Dg^2(y) \Delta y. \quad (4.44)$$

In order to simplify the equation (4.44) we introduce a new coordinate

$$u := \tilde{g}(\tilde{y})zg^{-1}(y) \quad (4.45)$$

for which the condition (4.43) reduces to $u = \iota(u^*)$, i.e.

$$u \in \tilde{p}\mathfrak{M}p \cap \mathcal{U}(\mathfrak{M}). \quad (4.46)$$

Substituting $z = \tilde{g}^{-1}(\tilde{y})ug(y)$ and

$$\Delta z = -\tilde{g}^{-1}(\tilde{y})D\tilde{g}(\tilde{y})\tilde{g}^{-1}(\tilde{y}) \Delta \tilde{y}ug(y) + \\ + \tilde{g}^{-1}(\tilde{y}) \Delta ug(y) + \tilde{g}^{-1}(\tilde{y})uDg(y) \Delta y \quad (4.47)$$

and

$$\Delta z^* = Dg(y) \Delta yu^*\tilde{g}^{-1}(\tilde{y}) + g(y) \Delta u^*g^{-1} + \\ -g(y)u^*\tilde{g}^{-1}(\tilde{y})D\tilde{g}(\tilde{y}) \Delta \tilde{y}g^{-1}(y) \quad (4.48)$$

into the equation (4.44) we obtain the equivalent equation

$$u^* \Delta u + (u^* \Delta u)^* = 0. \quad (4.49)$$

Thus for $x \in \Omega_{\tilde{p}p} \cap \mathcal{U}(\mathfrak{M})$ we have the isomorphisms of the real Banach spaces

$$T_x^+ \mathcal{G}(\mathfrak{M}) \cong (1 - \tilde{p})\mathfrak{M}\tilde{p} \oplus \mathfrak{H}_p \oplus (1 - p)\mathfrak{M}p \quad (4.50)$$

where the real Banach space

$$\mathfrak{H}_p = \{u^* \Delta u \in p\mathfrak{M}p : u^* \Delta u + (u^* \Delta u)^* = 0\} \quad (4.51)$$

is isomorphic with the space $ip\mathfrak{M}^h p$ of the anti-hermitian elements of the subalgebra $p\mathfrak{M}p$. Note here that fixing $u_0 \in \mathcal{U}(\mathfrak{M})_p^{\tilde{p}} \subset \mathcal{G}(\mathfrak{M})_p^{\tilde{p}}$ we obtain the bijection

$$\mathcal{U}(\mathfrak{M})_p^{\tilde{p}} \ni u \mapsto u_0^* u \in U(p\mathfrak{M}p). \quad (4.52)$$

The above allows us to define the one-to-one maps

$$\varphi_{\tilde{p}p} : \mathcal{U}(\mathfrak{M}) \cap \Omega_{\tilde{p}p} \rightarrow (1 - \tilde{p})\mathfrak{M}\tilde{p} \oplus ip\mathfrak{M}^h p \oplus (1 - p)\mathfrak{M}p \quad (4.53)$$

of $\mathcal{U}(\mathfrak{M}) \cap \Omega_{\tilde{p}p}$ on the open subset in the real Banach space $(1 - \tilde{p})\mathfrak{M}\tilde{p} \oplus ip\mathfrak{M}^h p \oplus (1 - p)\mathfrak{M}p$. Summing up the above facts we can formulate the following statement

Theorem 4.6. *(i) The groupoid $\mathcal{U}(\mathfrak{M})$ of partial isometries has a natural structure of the real Banach manifold of the type \mathfrak{G} , where the family \mathfrak{G} consist of the real Banach spaces*

$$(1 - \tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}^h p \oplus (1 - p)\mathfrak{M}p$$

parameterized by the pairs $(\tilde{p}, p) \in \mathcal{L}(\mathfrak{M}) \times \mathcal{L}(\mathfrak{M})$ of equivalent projections.

(ii) The groupoid $\mathcal{U}(\mathfrak{M})$ is a closed real Banach submanifold of $\mathcal{G}(\mathfrak{M})$ with the real Banach manifold structure underlying its complex Banach manifold structure.

We conclude from the Proposition 4.6 the $\mathcal{G}(\mathfrak{M})$ is the complexification of $\mathcal{U}(\mathfrak{M})$ in sense of the definition given in [3].

5 Groupoids and Banach Lie-Poisson structure of \mathfrak{M}_*

In [14] it was shown that the predual space \mathfrak{M}_* of W^* -algebra \mathfrak{M} has canonically defined Lie-Poisson structure. This follows from $ad^*(\mathfrak{M})$ -invariance of Banach subspace $\mathfrak{M}_* \subset \mathfrak{M}^*$, where $ad_x(y) := xy - yx$. One defines the Lie-Poisson bracket of $f, g \in C^\infty(\mathfrak{M}_*, \mathbb{C})$ as follows

$$\{f, g\} := \langle \omega, [Df(\omega), Dg(\omega)] \rangle. \quad (5.1)$$

Note that Frechét derivatives $Df(\omega), Dg(\omega)$ belong to \mathfrak{M} which allows to take the commutator of them. The predual space \mathfrak{M}_* as well as the Lie-Poisson bracket (5.1) is invariant with respect to the Ad^* -action of the Banach group $G(\mathfrak{M})$.

Multiplying the right hand side of definition (5.1) by $i = \sqrt{-1}$ one obtains the Lie-Poisson bracket for real valued functions $f, g \in C^\infty(\mathfrak{M}_*^h, \mathbb{R})$ defined on the hermitian part \mathfrak{M}_*^h of \mathfrak{M}_* . This follows from the paring between \mathfrak{M}_*^h and the real Banach-Lie algebra $i\mathfrak{M}^h$ of anti-hermitian elements of \mathfrak{M} defined by

$$\mathfrak{M}_*^h \times i\mathfrak{M}^h \ni (\omega, x) \mapsto i\langle \omega, x \rangle \in \mathbb{R}.$$

As in the complex case the Banach Lie-Poisson structure of \mathfrak{M}_*^h is $Ad^*(U(\mathfrak{M}))$ -invariant. The orbits of coadjoint representation of $U(\mathfrak{M})$ are weak symplectic manifold. Thus they give symplectic foliation of Banach Lie-Poisson space

$(\mathfrak{M}_*^h, \{\cdot, \cdot\})$, see [14]. More information concerning this interesting subject one can find in [2] and [14].

Now let us apply the definitions of the groupoids structures on the tangent bundle TG and cotangent bundle T^*G of a Lie group G , e.g. see [12], to the case of Banach-Lie group $G(\mathfrak{M})$. We will do this with some modification. Namely in our considerations we replace the cotangent bundle $T^*G(\mathfrak{M})$ by the pre-cotangent bundle $T_*G(\mathfrak{M})$ of $G(\mathfrak{M})$. Note that in the finite dimensional case the bundles T^*G and T_*G are canonically isomorphic. In our case the cotangent bundle $T^*G(\mathfrak{M})$, opposite to the pre-cotangent bundle $T_*G(\mathfrak{M})$ does not have the symplectic structure related to the Banach Lie-Poisson structure of \mathfrak{M}_*^h defined by (5.1).

The groupoid structure on $TG(\mathfrak{M})$ is defined as follows. The base of $TG(\mathfrak{M})$ is the tangent space $T_eG(\mathfrak{M})$ at the identity element $e \in G(\mathfrak{M})$. The source map $\mathbf{s} : TG(\mathfrak{M}) \rightarrow T_eG(\mathfrak{M})$ and the target map $\mathbf{t} : TG(\mathfrak{M}) \rightarrow T_eG(\mathfrak{M})$ are defined as follows

$$\mathbf{s}(a) := DL_{\pi(a)^{-1}}(\pi(a))a, \quad (5.2)$$

$$\mathbf{t}(a) := DR_{\pi(a)^{-1}}(\pi(a))a,$$

where $a \in TG(\mathfrak{M})$ and $\pi : TG(\mathfrak{M}) \rightarrow G(\mathfrak{M})$ is the canonical projection on the base. The identity section $\varepsilon : T_eG(\mathfrak{M}) \rightarrow TG(\mathfrak{M})$ is done by the inclusion of the fibre $T_eG(\mathfrak{M}) \subset TG(\mathfrak{M})$. The involution $\iota : TG(\mathfrak{M}) \rightarrow TG(\mathfrak{M})$ one defines by

$$\iota(a) := DL_{\pi(a)^{-1}}(e) \circ DR_{\pi(a)^{-1}}(\pi(a))a. \quad (5.3)$$

Finally the groupoid product is defined by

$$ab := DL_{\pi(a)}(\pi(b))b \quad (5.4)$$

if and only if $(a, b) \in TG(\mathfrak{M})^{(2)}$, i.e. $\mathbf{s}(a) = \mathbf{t}(b)$.

As a base for groupoid structure of $T_*G(\mathfrak{M})$ we assume the pre-cotangent bundle $T_{*e}G(\mathfrak{M})$ at $e \in G(\mathfrak{M})$. The identity section $\varepsilon_* : T_{*e}G(\mathfrak{M}) \rightarrow T_*G(\mathfrak{M})$ we define as an inclusion $T_{*e}G(\mathfrak{M}) \subset T_*G(\mathfrak{M})$.

Let us take $\xi \in T_*G(\mathfrak{M})$ and let $\pi_*(\xi) \in G(\mathfrak{M})$ be the projection of ξ on the base. Then one defines the source and target maps as follows:

$$\begin{aligned} \mathbf{s}_*(\xi) &:= (DL_{\pi(\xi)}(e))^*\xi, \\ \mathbf{t}_*(\xi) &:= (DR_{\pi(\xi)}(e))^*\xi, \end{aligned} \quad (5.5)$$

The inversion $\iota_* : T_*G(\mathfrak{M}) \rightarrow T_*G(\mathfrak{M})$ is defined by

$$\iota_*(\xi) := (DL_{\pi(\xi)}(\pi(\xi))^{-1})^* \circ (DR_{\pi(\xi)}(e))^*\xi. \quad (5.6)$$

The product of elements $\xi, \eta \in T_*G(\mathfrak{M})$ such that $\mathbf{s}(\xi) = \mathbf{t}(\eta)$ is given by

$$\xi\eta := (DL_{(\pi(\xi))^{-1}}(\pi(\xi)\pi(\eta)))^*\eta. \quad (5.7)$$

The precotangent bundle $T_*G(\mathfrak{M})$ is the weak symplectic complex Banach manifold with the weak symplectic form defined in the following way

$$\begin{aligned} \Omega_L(g, \rho)((a, \xi), (b, \eta)) &= \\ &= \langle \eta, DL_{g^{-1}}(g)a \rangle + \langle \xi, DL_{g^{-1}}(g)b \rangle - \langle \rho, [DL_{g^{-1}}(g)a, DL_{g^{-1}}(g)b] \rangle, \end{aligned} \quad (5.8)$$

where $g \in G(\mathfrak{M})$, $a, b \in T_gG(\mathfrak{M})$, $\rho, \xi, \eta \in T_{*e}G(\mathfrak{M})$, see [14]. Thus defined weak symplectic structure is consistent with the groupoid structure of $T_*G(\mathfrak{M})$ in the sense of [11], [20]. Hence one can consider $T_*G(\mathfrak{M})$ as a weak symplectic groupoid.

The definition of action groupoid structure on the product $G \times M$, where G is a group acting on a set M , one finds in Appendix D. From this general definition one gets action groupoid structures on $G(\mathfrak{M}) \times \mathfrak{M}$ and $G(\mathfrak{M}) \times \mathfrak{M}_*$ defined by adjoint $Ad : G(\mathfrak{M}) \rightarrow Aut \mathfrak{M}$ and co-adjoint $Ad^* : G(\mathfrak{M}) \rightarrow Aut \mathfrak{M}_*$ representation of Banach-Lie group $G(\mathfrak{M})$:

$$Ad_g x = gxg^{-1} \quad (5.9)$$

$$\langle Ad_g^* \omega, x \rangle := \langle \omega, Ad_{g^{-1}} x \rangle, \quad (5.10)$$

where $x \in \mathfrak{M}$ and $\omega \in \mathfrak{M}_*$ respectively.

The vector bundles trivializations $\phi : TG(\mathfrak{M}) \rightarrow G(\mathfrak{M}) \times \mathfrak{M}$ and $\phi_* : T_*G(\mathfrak{M}) \rightarrow G(\mathfrak{M}) \times \mathfrak{M}_*$ defined by

$$\phi(a) := (\pi(a), DL_{\pi(a)^{-1}}(\pi(a))a) \quad (5.11)$$

$$\phi_*(\xi) := (\pi(\xi), (DL_{\pi(\xi)}(e))^* \xi) \quad (5.12)$$

give the canonical groupoid isomorphisms $\phi : TG(\mathfrak{M}) \rightarrow G(\mathfrak{M}) \times \mathfrak{M}$ and $\phi_* : T_*G(\mathfrak{M}) \rightarrow G(\mathfrak{M}) \times \mathfrak{M}_*$. To this end we take the identifications $T_e U(\mathfrak{M}) \cong \mathfrak{M}$ and $T_{*e} G(\mathfrak{M}) \cong \mathfrak{M}_*$.

Let us define the injective immersions of the groupoids $\Lambda : TG(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M}) *_I \mathfrak{M}$ and $\Lambda_* : T_*G(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M}) *_I \mathfrak{M}_*$ by:

$$\Lambda(a) := (\pi(a) \, l(DL_{\pi(a)^{-1}}(\pi(a))a), DL_{\pi(a)^{-1}}(\pi(a))a) \quad (5.13)$$

$$\Lambda_*(\xi) := (\pi(\xi) \, l((DL_{\pi(\xi)}(e))^* \xi), DL_{\pi(\xi)}(e)^* \xi) \quad (5.14)$$

respectively.

In order to see that $\Lambda : TG(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M}) *_I \mathfrak{M}$ commutes with source and target maps we notice that

$$\begin{aligned} (\tilde{s} \circ \Lambda)(a) &= \tilde{s} \left(\pi(a) \, l(DL_{\pi(a)^{-1}}(\pi(a))a), DL_{\pi(a)^{-1}}(\pi(a))a \right) = \\ &= DL_{\pi(a)^{-1}}(\pi(a))a = (id \circ s)(a), \\ (\tilde{t} \circ \Lambda)(a) &= \tilde{t} \left(\pi(a) \, l(DL_{\pi(a)^{-1}}(\pi(a))a), DL_{\pi(a)^{-1}}(\pi(a))a \right) = \\ &= Ad_{\pi(a)} \, l(DL_{\pi(a)^{-1}}(\pi(a))a) DL_{\pi(a)^{-1}}(\pi(a))a = Ad_{\pi(a)} DL_{\pi(a)^{-1}}(\pi(a))a = (id \circ t)(a). \end{aligned}$$

Since $l(DL_{\pi(a)^{-1}}(\pi(a))a) = Ad_{\pi(a)}l(DL_{\pi(b)^{-1}}(\pi(b))b)$ the following shows that Λ preserves also the groupoid product

$$\begin{aligned} \Lambda(a)\Lambda(b) &= \\ &= (\pi(a) \ l(DL_{\pi(a)^{-1}}(\pi(a))a), DL_{\pi(a)^{-1}}(\pi(a))a) (\pi(b) \ l(DL_{\pi(b)^{-1}}(\pi(b))b), DL_{\pi(b)^{-1}}(\pi(b))b) = \\ &= ((\pi(a) \ l(DL_{\pi(a)^{-1}}(\pi(a))a)(\pi(b)) \ l(DL_{\pi(b)^{-1}}(\pi(b))b), DL_{\pi(b)^{-1}}(\pi(b))b) = \\ &= \Lambda(DL_{\pi(a)}(\pi(b))b) = \Lambda(ab). \end{aligned}$$

Summarizing the above facts we obtain the following groupoid monomorphism

$$\begin{array}{ccc} TG(\mathfrak{M}) & \xrightarrow{\Lambda} & \mathcal{G}(\mathfrak{M}) *_l \mathfrak{M} \\ \begin{array}{c} \downarrow \quad \downarrow \\ \mathbf{s} \quad \mathbf{t} \end{array} & & \begin{array}{c} \downarrow \quad \downarrow \\ \tilde{\mathbf{s}} \quad \tilde{\mathbf{t}} \end{array} \\ \mathfrak{M} & \xrightarrow{id} & \mathfrak{M} \end{array} \quad (5.15)$$

In a similar way we obtain the monomorphism

$$\begin{array}{ccc} T_*G(\mathfrak{M}) & \xrightarrow{\Lambda_*} & \mathcal{G}(\mathfrak{M}) *_l \mathfrak{M}_* \\ \begin{array}{c} \downarrow \quad \downarrow \\ \mathbf{s} \quad \mathbf{t} \end{array} & & \begin{array}{c} \downarrow \quad \downarrow \\ \tilde{\mathbf{s}} \quad \tilde{\mathbf{t}} \end{array} \\ \mathfrak{M}_* & \xrightarrow{id} & \mathfrak{M}_* \end{array} \quad (5.16)$$

of the predual groupoids $T_*\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathfrak{M}_*$ into the action groupoid $\mathcal{G}(\mathfrak{M}) *_l \mathfrak{M}_* \rightrightarrows \mathfrak{M}_*$

Now instead of complex Banach-Lie group $G(\mathfrak{M})$ let us consider the groupoid of partially invertible elements $\mathcal{G}(\mathfrak{M})$. In this case we come to the following statements.

The tangent prolongation $T\mathcal{G}(\mathfrak{M}) \rightrightarrows T\mathcal{L}(\mathfrak{M})$ of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is a Banach Lie \mathcal{VB} -groupoid, (see e.g. [13]), i.e. one has

$$\begin{array}{ccc} T\mathcal{G}(\mathfrak{M}) & \xrightarrow{\tilde{q}} & \mathcal{G}(\mathfrak{M}) \\ \begin{array}{c} \downarrow \quad \downarrow \\ D\mathbf{s} \quad D\mathbf{t} \end{array} & & \begin{array}{c} \downarrow \quad \downarrow \\ \mathbf{s} \quad \mathbf{t} \end{array} \\ T\mathcal{L}(\mathfrak{M}) & \xrightarrow{q} & \mathcal{L}(\mathfrak{M}) \end{array} \quad (5.17)$$

where the bundle vector projections q and \tilde{q} define the groupoid morphism, the tangent maps $Ds, Dt, D\iota, D\varepsilon$ are vector bundle morphisms, and the map

$$(Dq, Ds) : T\mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M}) * T\mathcal{L}(\mathfrak{M})$$

of tangent groupoid $T\mathcal{G}(\mathfrak{M})$ on the action groupoid $\mathcal{G}(\mathfrak{M}) * T\mathcal{L}(\mathfrak{M})$ is a surjective submersion.

The core of $T\mathcal{G}(\mathfrak{M})$ is the algebroid $\mathcal{AG}(\mathfrak{M})$ of the groupoid $\mathcal{G}(\mathfrak{M})$. See e.g. [13] for the definition of the core of \mathcal{VB} -groupoid. The algebroid $\mathcal{AG}(\mathfrak{M})$ and its predual $\mathcal{A}_*\mathcal{G}(\mathfrak{M})$ are most crucial for the Poisson aspect of the investigated theory. Namely, extending the considerations from the finite dimensional case to the Banach-Lie context we obtain the Banach-Lie \mathcal{VB} -groupoid

$$\begin{array}{ccc} T_*\mathcal{G}(\mathfrak{M}) & \xrightarrow{\tilde{q}_*} & \mathcal{G}(\mathfrak{M}) \\ \tilde{s} \downarrow \downarrow \tilde{t} & & s \downarrow \downarrow t \\ \mathcal{A}_*(\mathfrak{M}) & \xrightarrow{q_*} & \mathcal{L}(\mathfrak{M}) \end{array} \quad (5.18)$$

precotangent to the one presented in (5.17), where q_* and \tilde{q}_* are the projections on the base. One defines the source \tilde{s} and target \tilde{t} maps in (5.18) as follows. Let $\phi \in T_{*x}\mathcal{G}(\mathfrak{M})$ and $x \in \mathcal{A}_p\mathcal{G}(\mathfrak{M})$, $p \in \mathcal{L}(\mathfrak{M})$, then

$$\langle \tilde{s}(\phi), x \rangle := \langle \phi, DL_g(\varepsilon(p))(x - D\varepsilon(p)Dt(\varepsilon(p))x) \rangle, \quad (5.19)$$

$$\langle \tilde{t}(\phi) \rangle := \langle \phi, DR_g(\varepsilon(p))x \rangle. \quad (5.20)$$

The product $\phi \bullet \psi$ of $\phi \in T_{*x}\mathcal{G}(\mathfrak{M})$ and $\psi \in T_{*y}\mathcal{G}(\mathfrak{M})$, where $\tilde{s}(\phi) = \tilde{t}(\psi) \in \mathcal{A}_{*p}\mathcal{G}(\mathfrak{M})$ and $s(x) = t(y) = p \in \mathcal{L}(\mathfrak{M})$, one defines by

$$\langle \phi \bullet \psi, \xi \cdot \eta \rangle = \langle \phi, \xi \rangle + \langle \psi, \eta \rangle, \quad (5.21)$$

where $\xi \in T_x\mathcal{G}(\mathfrak{M})$, $\eta \in T_y\mathcal{G}(\mathfrak{M})$ satisfy $Ds(\xi) = Dt(\eta)$ and $\xi \cdot \eta \in T_{xy}\mathcal{G}(\mathfrak{M})$ is the product of ξ and η in the tangent groupoid $T\mathcal{G}(\mathfrak{M})$. The above definitions we obtain as a direct generalization of those accepted in the finite dimensional case, e.g. [13].

The groupoid $T_*\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{A}_*\mathcal{G}(\mathfrak{M})$ is a weak symplectic Banach-Lie realization of the Banach-Poisson bundle $\mathcal{A}_*\mathcal{G}(\mathfrak{M})$, which Poisson structure is determined by the algebroid structure of $\mathcal{AG}(\mathfrak{M})$. We note here that diagram (5.18) is the groupoid version of the diagram

$$\begin{array}{ccc} T_*G(\mathfrak{M}) & \xrightarrow{\tilde{q}_*} & G(\mathfrak{M}) \\ \tilde{s} \downarrow \downarrow \tilde{t} & & s \downarrow \downarrow t \\ \mathfrak{M}_* & \xrightarrow{q_*} & \{1\} \end{array} \quad (5.22)$$

valid for the group $G(\mathfrak{M})$.

The proofs of these statements are the direct generalizations of the proofs for the finite dimensional case to the context of the Banach-Lie groupoids theory.

Finally let us mention that all objects considered above belong to the category of complex analytic Banach manifold. They have their real analytic counterparts if we replace the group $G(\mathfrak{M})$ and the groupoid $\mathcal{G}(\mathfrak{M})$ by $U(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M})$ respectively, and \mathfrak{M} (\mathfrak{M}_*) by $i\mathfrak{M}^h$ (\mathfrak{M}_*^h).

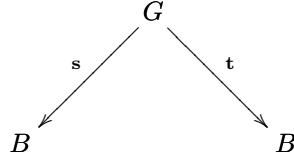
Authors apologize for the fact that subjects discussed in this section are treated in the abbreviated way. However, the detailed investigation of Banach-Lie Poisson geometry related to W^* -algebras needs longer treatment in a separate paper, which is currently in preparation.

6 Appendix

A Groupoid

Let us recall that a **groupoid with the base set** B (set of objects) is a set G such that:

- i) there is a pair of maps



called **source** and **target** map respectively;

- ii) for set of composable pairs

$$G^{(2)} := \{(g, h) \in G \times G; \quad \mathbf{s}(g) = \mathbf{t}(h)\}$$

one has a **product map** $m : G^{(2)} \rightarrow G$, denoted by

$$m(g, h) =: gh \tag{6.1}$$

such that

- (a) $\mathbf{s}(gh) = \mathbf{s}(h), \quad \mathbf{t}(gh) = \mathbf{t}(g),$
- (b) associativity: $k(gh) = (kg)h;$

- iii) there is an injection $\varepsilon : B \rightarrow G$ called the **identity section**, such that

$$\varepsilon(\mathbf{t}(g))g = g = g\varepsilon(\mathbf{s}(g));$$

iv) there exists an **inversion** $\iota : G \rightarrow G$ denoted by

$$\iota(g) =: g^{-1}, \quad (6.2)$$

such that

$$\iota(g)g = \varepsilon(\mathbf{s}(g)), \quad g\iota(g) = \varepsilon(\mathbf{t}(g))$$

for all $g \in G$.

A groupoid G gives rise to a hierarchy of sets

$$G^{(0)} := \varepsilon(B) \simeq (B)$$

$$G^{(1)} := G$$

$$G^{(2)} := \{(g, h) \in G \times G; \quad \mathbf{s}(g) = \mathbf{t}(h)\}$$

\vdots

$$G^{(k)} = \{(g_1, g_2, \dots, g_k) \in G \times G \times \dots \times G; \quad \mathbf{t}(g_i) = \mathbf{s}(g_{i-1}), \quad i = 2, 3, \dots, k\}$$

In the paper we will consider the topological (differentiable) groupoids. Because of this let us recall that the groupoid G is called a topological (differentiable) groupoid if G and B have the topologies (differential manifold structure) such that:

- i) the product map (6.1) and the involution (6.2) are continuous (differentiable);
- ii) the injection $\varepsilon : B \rightarrow G$ is an embedding (differentiable embedding).

From $\mathbf{s}(g) = \varepsilon^{-1}(gg^{-1})$ and $\mathbf{t}(g) = \varepsilon^{-1}(g^{-1}g)$ it follows that source map and target map are continuous (differentiable). By definition the topology of $G^{(k)}$, for $k = 0, 1, 2, \dots$, is inherited from G . In case of differentiable groupoid one assumes that the source and target maps are submersions.

B Groupoids morphism

A **morphism** ϕ of two groupoids G_1 and G_2 over bases B_1 and B_2 one can depict by the following commutative diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{\phi_G} & G_2 \\ \mathbf{s}_1 \downarrow \downarrow \mathbf{t}_1 & & \mathbf{s}_2 \downarrow \downarrow \mathbf{t}_2 \\ B_1 & \xrightarrow{\phi_B} & B_2 \end{array} \quad (6.3)$$

By definition one also has

$$\phi_B \circ \mathbf{s}_1 = \mathbf{s}_2 \circ \phi_G \quad \text{and} \quad \phi_B \circ \mathbf{t}_1 = \mathbf{t}_2 \circ \phi_G$$

and

$$\phi_G(g)\phi_G(h) = \phi_G(gh)$$

for $gh \in G_1^{(2)}$. If $\phi_G : G_1 \hookrightarrow G_2$ and $\phi_B : B_1 \hookrightarrow B_2$ are inclusion maps one says that G_1 is **subgroupoid** of G_2 . The subgroupoid $G_1 \subset G_2$ is a **wide subgroupoid** of G_2 if $\mathbf{s}_1(G_1) = \mathbf{t}_1(G_1) = B_2$.

An example of groupoid morphism is given by

$$\begin{array}{ccc}
 G & \xrightarrow{(\mathbf{s}, \mathbf{t})} & B \times B \\
 \mathbf{s} \downarrow \quad \downarrow \mathbf{t} & & \downarrow pr_1 \quad \downarrow pr_2 \\
 B & \xrightarrow{id} & B
 \end{array} \tag{6.4}$$

where $B \times B$ is the pair groupoid, i.e. $\mathbf{s} := pr_1$, $\mathbf{t} := pr_2$, $\iota(x, y) := (y, x)$, $\varepsilon(x) = (x, x)$ and $m((y, z), (x, y)) = (x, z)$.

C Action of groupoid

We recall the definition of the **left action of groupoid G on the set M** . One assumes for this reason that there exists a map (moment map)

$$\mu : M \rightarrow B \tag{6.5}$$

and one defines the space

$$G *_l M := \{(g, r) \in G \times M : \mathbf{s}(g) = \mu(r)\}. \tag{6.6}$$

Then the left action of groupoid G on M is defined as a map $G * M \ni (g, r) \mapsto g \cdot r \in M$ with properties:

$$\begin{aligned}
 (gh) \cdot r &= g \cdot (h \cdot r) \\
 \mu(g \cdot r) &= \mathbf{t}(g) \\
 \varepsilon(\mu(r)) \cdot r &= r.
 \end{aligned} \tag{6.7}$$

For the **right action of G on M** instead of (6.7) we have

$$\begin{aligned}
 r \cdot (gh) &= (r \cdot h) \cdot g \\
 \mu(r \cdot g) &= \mathbf{s}(g) \\
 r \cdot \varepsilon(\mu(r)) &= r,
 \end{aligned} \tag{6.8}$$

where $(g, r) \in G *_r M := \{(g, r) \in G \times M : \mathbf{t}(g) = \mu(r)\}$.

As an example let us take the canonical left action of G on its base B . In this case $M := B$, $\mu := id$ and

$$G * B = \{(g, x) : x = \mathbf{s}(g)\} \tag{6.9}$$

The action map is defined by

$$G * B \ni (g, x) \mapsto g \cdot x := \mathbf{t}(g). \tag{6.10}$$

The defining properties (6.7) follow from the corresponding properties of the maps \mathbf{s}, \mathbf{t} , ε and the product map (6.1).

One equips the set $\tilde{G} := G * M$ with the groupoid structure defined as follows:

i) source map and target map are given by $\tilde{s}(g, r) := r \in M$ and $\tilde{t}(g, r) := g \cdot r \in M$;

ii) the set of composable pairs

$$\tilde{G}^{(2)} := \{((g, r), (h, n)) \in \tilde{G} \times \tilde{G}; \quad \mathbf{t}(h) = \mathbf{s}(g)\}$$

and the product map $\tilde{m} : \tilde{G}^{(2)} \rightarrow \tilde{G}$ is defined as

$$\tilde{m}((g, r), (h, n)) = (gh, n); \quad (6.11)$$

iii) the identity section $\tilde{\varepsilon} : M \rightarrow \tilde{G}$ is defined by

$$\tilde{\varepsilon}(r) = (\varepsilon(\mu(r)), r); \quad (6.12)$$

iv) the involution $\tilde{\iota} : \tilde{G} \rightarrow \tilde{G}$ is defined by

$$\tilde{\iota}(g, r) = (\iota(g), g \cdot r). \quad (6.13)$$

In the case when G is a topological groupoid and M is a topological space we obtain on \tilde{G} the structure of the topological groupoid if the moment map μ and the action G on M are continuous. The topological structure of $\tilde{G} \subset G \times M$ is inherited from product topology of $G \times M$.

One calls the morphism depicted in (6.3) a **covering morphism** if for each $x \in B_1$ the restriction $\phi_G : \mathbf{s}^{-1}(x) \rightarrow \mathbf{s}^{-1}(\phi_B(x))$ of ϕ_G to the \mathbf{s} -level of x is bijection.

The diagram

$$\begin{array}{ccc} G *_l M & \xrightarrow{pr_1} & G \\ \tilde{s} \downarrow \downarrow \tilde{t} & & \downarrow \downarrow s \quad t \\ M & \xrightarrow{\mu} & B \end{array} \quad (6.14)$$

where

$$\phi_G(g, r) := pr_1(g, r) = g \quad \text{and} \quad \phi_B(r) := \mu(r),$$

gives an example of the covering morphism of groupoids.

D Action groupoid

If a group G acts on a set M

$$G \times M \ni (g, m) \mapsto g \cdot m \in M$$

one can define on the set $G \times M$ the groupoid structure, which is called **action groupoid** structure. For this case one defines

i) source and target maps $\mathbf{s}', \mathbf{t}' : G \times M \rightarrow M$ as

$$\mathbf{s}'(g, m) := m \in M \quad \text{and} \quad \mathbf{t}'(g, m) := g \cdot m ; \quad (6.15)$$

ii) the groupoid product

$$(g, m)(h, n) := (gh, n) \quad (6.16)$$

on the set of composable pairs

$$(G \times M)^{(2)} := \{((g, m), (h, n)) \in (G \times M) \times (G \times M) : m = h \cdot n\}$$

iii) the identity section $\varepsilon' : M \rightarrow G \times M$ by

$$\varepsilon'(m) = (e, m); \quad (6.17)$$

iv) the involution $\iota' : G \times M \rightarrow G \times M$ by

$$\iota'(g, m) = (g^{-1}, g \cdot m). \quad (6.18)$$

E Bisections of groupoid

By a **bisection** of a groupoid G one understands a subset σ such that the restrictions $\mathbf{s} : \sigma \rightarrow B$ and $\mathbf{t} : \sigma \rightarrow B$ of the source map and target map to σ are bijections of σ on B . The set $\mathfrak{B}(G)$ of all bisections of the groupoid G form the group if one defines the product of two bisections σ_1 and σ_2 as follows:

$$\sigma_1 \circ \sigma_2 := \{gh \in G; (g, h) \in (\sigma_1 \times \sigma_2) \cap G^{(2)}\} \quad (6.19)$$

The identity element of the **bisection group** $(\mathfrak{B}(G), \circ)$ is just the identity section $B \cong G^{(0)}$. The map

$$\mathfrak{B}(G) \ni \sigma \rightarrow (\mathbf{t}|_{\sigma}) \circ (\mathbf{s}|_{\sigma})^{-1} \in \text{Bij } B$$

is the group monomorphism. Therefore one can consider $\mathfrak{B}(G)$ as a subgroup of the group $\text{Bij } B$ of bijections of B .

A **local bisection** of a groupoid G is a subset $\sigma \subset G$ such that $\mathbf{s} : \sigma \rightarrow B$ and $\mathbf{t} : \sigma \rightarrow B$ are injections on σ into B . The maps

$$\mathbf{t} \circ \mathbf{s}^{-1} : \mathbf{s}(\sigma) \rightarrow \mathbf{t}(\sigma) \quad \text{and} \quad \mathbf{s} \circ \mathbf{t}^{-1} : \mathbf{t}(\sigma) \rightarrow \mathbf{s}(\sigma) \quad (6.20)$$

are **partial bijections** of B . They define the inverse subsemigroup $\mathfrak{B}_{loc}(G) \subset B_{part}(B)$ of the inverse semigroup $B_{part}(B)$ of all partial bijections of B . One calls $\mathfrak{B}_{loc}(G)$ the inverse semigroup of local bisections. Let us recall that the semigroup product in $B_{part}(B)$ is just the superposition of partial bijections.

Let us remark a motivation to use terms "bisection" and "local bisection" is that $(\mathbf{s}|_{\sigma})^{-1} : B \rightarrow \sigma \subset G$ ($(\mathbf{t}|_{\sigma})^{-1} : B \rightarrow \sigma \subset G$) and $(\mathbf{s}|_{\sigma})^{-1} : \mathbf{s}(\sigma) \rightarrow \sigma \subset G$ ($(\mathbf{t}|_{\sigma})^{-1} : \mathbf{t}(\sigma) \rightarrow \sigma \subset G$) are section and local section of the bundle $\mathbf{s} : G \rightarrow B$ ($\mathbf{t} : G \rightarrow B$).

ACKNOWLEDGEMENTS

We thank Kirill Mackenzie for inspiring discussions within the Lie \mathcal{VB} -groupoids and Lie algebroids. We also thank Daniel Beltita for useful remarks which made the paper transparent.

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